Brownian Motion

2022-2023 Academic year Semester 2

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Notice This course seems to experience a major change, and the following note will not completely follows the syllabus. However, this still roughly covers all the things we developed during the course.

1 Preliminaries

1.1 Probability and Expectations

▶ **Definition 1.1** (σ-algebra)**.** A collection $\mathcal{F}$ of subsets of Ω is called σ-algebra if the following three conditions holds

- $\Omega \in \mathcal{F}$.
- If $A \in \mathcal{F}$, then $A^c := \Omega \setminus A \in \mathcal{F}$, and
- If $A_i \in \mathcal{F}$ for $i = 1, 2, \dots$, then $\bigcup_{i=1}^{\infty} A_i = A_1 \cup A_2 \cup \dots \in \mathcal{F}$.

Let Ω be the sample space and $\mathcal{F}$ be a σ-algebra on Ω, we call the tuple $(\Omega, \mathcal{F})$ a *measurable space.*

▶ **Definition 1.2.** For any collection $\mathcal{D}$ of subsets of Ω, the smallest σ-algebra $\mathcal{G}$ that contains all elements of $\mathcal{D}$ is called the σ-algebra generated by $\mathcal{D}$. We write $\mathcal{G} = \sigma(\mathcal{D})$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

▶ **Definition 1.3** (random variable)**.** Let $X : \Omega \mapsto \mathbb{R}$ be a function on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then, X is a random variable if for any $B \in \mathcal{B}(\mathbb{R})$, the pre-image

$$X^{-1}(B) = \{\omega \in \Omega \colon X(\omega) \in B\} \in \mathcal{F}.$$

We can then define the probability law on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, $\mathbb{P}_X$, defined by

$$\mathbb{P}_X(A) = \mathbb{P}(X^{-1}(A)), \qquad \forall A \in \mathcal{B}(\mathbb{R}).$$

Then, two random variables X and Y have the same *law* if

$$\mathbb{P}_X(B) = \mathbb{P}_Y(B), \qquad \forall B \in \mathcal{B}(\mathbb{R}).$$

If X and Y have the same law, we write $X \overset{d}{=} Y$.

▶ **Definition 1.4.** Let X be a random variable. We define $\sigma(X)$, the σ-algebra generated by X, as the minimal σ-algebra with respect to which X is measurable, that is

$$\sigma(X) = \big\{\{\omega \colon X(\omega) \in B\},\ B \in \mathcal{B}(\mathbb{R})\big\}.$$

▶ **Definition 1.5** (Borel function)**.** A function $g : \mathbb{R} \to \mathbb{R}$ is a Borel function if $g^{-1}(B) \in \mathcal{B}(\mathbb{R})$ for any $B \in \mathcal{B}(\mathbb{R})$.

▶ **Lemma 1.6.** *Let X and Y be random variables. If Y be a random variable which is $\sigma(X)$-measurable. Then there exists a Borel function $g : \mathbb{R} \to \mathbb{R}$, such that $Y(\omega) = g(X(\omega))$, $\omega \in \Omega$.*

For $A \in \mathcal{F}$, define the indicator function of the event A as

$$\mathbf{1}_A(\omega) = \begin{cases} 1, & \text{if } \omega \in A, \\ 0, & \text{if } \omega \in A^c. \end{cases}$$

We will now define the expectation of a random variable X against a probability measure, denoted as $\mathbb{E}[X]$ or $\int_\Omega X(\omega) \mathrm{d}\mathbb{P}(\omega)$.

If X is a *discrete random variable* that takes values in the finite set $R_X = \{x_1, x_2, \dots, x_n\}$, we can present it as follows

$$X(\omega) = \sum_{i=1}^{n} x_i \mathbf{1}_{A_i}(\omega),$$

where $A_i = \{\omega \colon X(\omega) = x_i\}$. Then we can define the expectation as

$$\mathbb{E}[X] = \sum_{i=1}^{n} x_i \mathbb{P}(A_i) = \sum_{i=1}^{n} x_i \mathbb{P}(X = x_i).$$

For a continuous random variable X with the probability density function $f_X(x)$, we have

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x) \mathrm{d}x.$$

The indicator function have the following property,

$$\int_{\Omega} \mathbf{1}_A(\omega) \mathrm{d}\mathbb{P}(\omega) = \mathbb{E}[\mathbf{1}_A] = \mathbb{P}(A).$$

In general, we will write the expectation of X on the event A as

$$\mathbb{E}[X; A] = \int_A X(\omega) \mathrm{d}\mathbb{P}(\omega) = \int_{\Omega} X(\omega) \mathbf{1}_A(\omega) \mathrm{d}\mathbb{P}(\omega) = \mathbb{E}[X \mathbf{1}_A].$$

▶ **Theorem 1.7** (properties of expectation)**.**

(1) *Linearity. If X and Y are integrable random variables, then the linear combination $\alpha X + \beta Y$ is integrable, and*

$$\mathbb{E}[\alpha X + \beta Y] = \alpha\, \mathbb{E}[X] + \beta\, \mathbb{E}[Y], \qquad \alpha, \beta \in \mathbb{R}.$$

(2) $|\, \mathbb{E}[X] \,| \leq \mathbb{E}[|\, X \,|]$.

(3) *Monotonicity. If $X \geq Y$, a.s., then $\mathbb{E}[X] \geq \mathbb{E}[Y]$.*

(4) *If $X \geq 0$, a.s., then $\mathbb{E}[X] \geq 0$ a.s.. Moreover, if $\mathbb{E}[X] = 0$, then $X = 0$ a.s..*

(5) *Jensen's inequality. If X is integrable, and g is a convex function, then*

$$\mathbb{E}[g(X)] \geq g(\mathbb{E}[X]).$$

(6) *If X and Y are integrable independent random variables, then XY is integrable, and $\mathbb{E}[XY] = \mathbb{E}[X]\, \mathbb{E}[Y]$.*

▶ **Theorem 1.8** (monotone convergence theorem)**.** *Let X_n be an increasing sequence of random variables bounded below by an integrable random variable Y and converges to a random variable X, that is $Y \leq X_n$ and $X_n \uparrow X$, a.s.. Then,*

$$\lim_{n \to \infty} \mathbb{E}[X_n] = \mathbb{E}[X].$$

▶ **Theorem 1.9** (dominated convergence theorem)**.** *Let X_n be a sequence of random variables converging to a random variable X, a.s.. Suppose that there exists an integrable random variable Y such that $|X_n(\omega)| \leq Y(\omega)$ a.s. for all $n \geq 1$. Then*

$$\lim_{n \to \infty} \mathbb{E}[X_n] = \mathbb{E}[X].$$

▶ **Definition 1.10** (conditional expectation)**.** Let X be an integrable random variable and $\mathcal{G}$ be a σ-algebra. A random variable Y is called the *conditional expectation* of X given $\mathcal{G}$, denoted as $Y = \mathbb{E}[X \mid \mathcal{G}]$, if

(A) Y is $\mathcal{G}$-measurable.

(B) For any event $A \in \mathcal{G}$, we have

$$\mathbb{E}[Y \mathbf{1}_A] = \mathbb{E}[\mathbb{E}[X \mid \mathcal{G}] \mathbf{1}_A] = \mathbb{E}[X \mathbf{1}_A].$$

▶ **Proposition 1.11** (properties of conditional expectation)**.**

(1) *Linearity. $\mathbb{E}[\alpha X_1 + \beta X_2 \mid \mathcal{G}] = \alpha\, \mathbb{E}[X_1 \mid \mathcal{G}] + \beta\, \mathbb{E}[X_2 \mid \mathcal{G}]$ for $\alpha, \beta \in \mathbb{R}$.*

(2) *If $X \geq 0$, a.s., then $\mathbb{E}[X \mid \mathcal{G}] \geq 0$.*

(3) *Conditional mean formula. $\mathbb{E}[\mathbb{E}[X \mid \mathcal{G}]] = \mathbb{E}[X]$, namely, one can freely introducing σ-algebra using expectations.*

(4) *Taking out what is known. If Z is $\mathcal{G}$-measurable random variable, then*

$$\mathbb{E}[ZX \mid \mathcal{G}] = Z\,\mathbb{E}[X \mid \mathcal{G}].$$

In particular, $\mathbb{E}[Z \mid \mathcal{G}] = Z$.

(5) *Role of independence. If X is independent of $\mathcal{G}$, then* $\mathbb{E}[X \mid \mathcal{G}] = \mathbb{E}[X]$.

(6) *Tower property. If $\mathcal{G}_1$ is another σ-algebra such that $\mathcal{G}_1 \subset \mathcal{G}$, then*

$$\mathbb{E}[\mathbb{E}[X \mid \mathcal{G}] \mid \mathcal{G}_1] = \mathbb{E}[X \mid \mathcal{G}_1].$$

In particular, if $\mathcal{G}_1 = \{\emptyset, \Omega\}$, then we obtain the condition mean formula (3).

(7) *Conditional Jensen's inequality. If X is integrable and g is a convex function, then*

$$\mathbb{E}[g(X) \mid \mathcal{G}] \geq g(\mathbb{E}[X \mid \mathcal{G}]).$$

► **Lemma 1.12** (Independence lemma)**.** *Let X and Y be random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathcal{X}$ and $\mathcal{Y}$ be σ-algebras. Assume that X is $\mathcal{X}$ measurable, Y is $\mathcal{Y}$ measurable and $\mathcal{X}$ is independent of $\mathcal{Y}$. Then, for all bounded functions $\Phi(\cdot,\cdot)$,*

$$\mathbb{E}[\Phi(X,Y) \mid \mathcal{X}] = \mathbb{E}[\Phi(x,Y)]|_{x=X} = \mathbb{E}[\Phi(X,Y) \mid X].$$

1.2 Gaussian Random Vectors

A continuous random variable W is defined to have *standard normal distribution* if its density is given by

$$f_W(w) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{w^2}{2}\right), \qquad w \in \mathbb{R}.$$

A random variable $X = \sigma W + \mu$ for arbitrary μ and $\sigma \geq 0$ is defined to have *normal distribution* if its density is given by

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \qquad x \in \mathbb{R}.$$

we say X is a Gaussian (normal) random variable and denote $X \sim N(\mu, \sigma^2)$.

► **Theorem 1.13.** *Let $X \sim N(\mu, \sigma^2)$, then*

(a) *The moment generating function of X is given by*

$$M_X(t) := \mathbb{E}[\mathrm{e}^{tX}] = \mathrm{e}^{\mu t + \sigma^2 t^2/2}, \qquad t \in \mathbb{R}.$$

(b) *The characteristic function of X is given by*

$$\varphi_X(t) := \mathbb{E}[\mathrm{e}^{\mathrm{i}tX}] = \mathrm{e}^{\mathrm{i}\mu t - \sigma^2 t^2/2}, \qquad t \in \mathbb{R}.$$

(c) *Mean and variance of the X are given by*

$$\mathbb{E}[X] = \mu, \quad \mathbb{V}[X] = \sigma^2.$$

Using the moment generating functions, one can show that the sum of independent normal random variables in normal as well. Namely, let

$$X_i \sim N(\mu_i, \sigma_i^2), \qquad i = 1, \ldots, n,$$

then for $r_1, \ldots, r_n \in \mathbb{R}$,

$$r_1 X_1 + \cdots + r_n X_n \sim N(r_1\mu_1 + \cdots + r_n\mu_n, r_1^2\sigma_1^2 + \cdots + r_k^2\sigma_k^2).$$

Consider a vector $\boldsymbol{W} = (W_1, \ldots, W_n)^T$, where W_i are i.i.d. random variables. Let $\boldsymbol{w} = (w_1, \ldots, w_n)^T$, then $\boldsymbol{W}$ is a multivariate normal (Gaussian) distributed if its probability density function is

$$f_{\boldsymbol{W}}(\boldsymbol{w}) = \frac{1}{(2\pi)^{n/2}} \exp\left(-\frac{\boldsymbol{w}^T\boldsymbol{w}}{2}\right).$$

▶ **Definition 1.14.** A random vector $\boldsymbol{X} = (X_1, \dots, X_n)^T$ has normal (Gaussian) distribution if it can be represented as

$$\boldsymbol{X} = A\boldsymbol{W} + \boldsymbol{\mu},$$

where $\boldsymbol{W} = (W_1, \dots, W_m)^T$ is a vector of i.i.d. standard normal random variables, $\boldsymbol{\mu} = (\mu_1, \dots, \mu_n)^T$ is a real vector and $A = (a_{ij})_{1\le i\le n,\ 1\le j\le m}$ is a matrix of size $n \times m$.

▶ **Theorem 1.15.** *Let $\boldsymbol{X} = (X_1, \dots, X_n)^T$ be a zero-mean Gaussian vector. Let $\boldsymbol{Y} = (Y_1, \dots, Y_m)^T = B\boldsymbol{X}$ for some $B \in \mathbb{R}^{m\times n}$. Then Y is a zero-mean Gaussian vector as well.*

An important characterization of a Gaussian vector $\boldsymbol{X} = (X_1, \dots, X_n)^T$ is its covariance matrix defined as

$$\mathrm{Cov}(\boldsymbol{X}) = (K_{ij})_{i,j=1}^n, \qquad K_{ij} = \mathbb{E}[(X_i - \mathbb{E}[X_i])(X_j - \mathbb{E}[X_j])].$$

Therefore we can rewrite the matrix K as

$$K = \mathbb{E}[(\boldsymbol{X} - \mathbb{E}[\boldsymbol{X}])(\boldsymbol{X} - \mathbb{E}[\boldsymbol{X}])^T].$$

If $\boldsymbol{X} = A\boldsymbol{W} + \mu$ where $\boldsymbol{W} = (W_1, \dots, W_n)^T$ and W_i are i.i.d. $N(0,1)$ random variables, then the covariance of $\boldsymbol{X}$ is $K = AA^T$.

▶ **Theorem 1.16.** *Let $\boldsymbol{X}$ be a Gaussian vector with the mean vector $\boldsymbol{\mu} = (\mu_1, \dots, \mu_n)^T$, and the covariance matrix $K \in \mathbb{R}^{n\times n}$. Then the moment generating function of $\boldsymbol{X}$ is given by*

$$M_{\boldsymbol{X}}(\boldsymbol{t}) = \exp\left(\boldsymbol{t}^T\boldsymbol{\mu} + \frac{\boldsymbol{t}^T K \boldsymbol{t}}{2}\right), \qquad \boldsymbol{t} = (t_1, \dots, t_n)^T \in \mathbb{R}^n,$$

and the characteristic function of $\boldsymbol{X}$ is given by

$$\varphi(\boldsymbol{\theta}) = \exp\left(\mathrm{i}\boldsymbol{\theta}\boldsymbol{\mu} - \frac{\boldsymbol{\theta}^T K \boldsymbol{\theta}}{2}\right), \qquad \boldsymbol{\theta} = (\theta_1, \dots, \theta_n)^T \in \mathbb{R}^n.$$

▶ **Theorem 1.17.** *Let $\boldsymbol{Z} = A\boldsymbol{W} + \boldsymbol{\mu}$, where $\boldsymbol{W}$ has standard multivariate normal distribution with nonsingular covariance matrix K. Then the probability density function of $\boldsymbol{Z}$ is given by*

$$f_{\boldsymbol{Z}}(\boldsymbol{z}) = \frac{1}{(2\pi)^{n/2}\sqrt{\det(K)}} \exp\left(-\frac{1}{2}(\boldsymbol{z} - \boldsymbol{\mu})^T K^{-1}(\boldsymbol{z} - \boldsymbol{\mu})\right), \qquad \boldsymbol{z} = (z_1, \dots, z_n)^T.$$

▶ **Proposition 1.18.** *Let $\boldsymbol{X} = (X_1, \dots, X_n)^T$ be a Gaussian vector. Then X_i are independent with each other if and only if they are uncorrelated. Namely, $\mathrm{Cov}(X_i, X_j) = 0$ for any $i \neq j$.*

▶ **Proposition 1.19** (Gaussian vector)**.** *Let $\boldsymbol{Z} = (Z_1, \dots, Z_n)^T$ be a random vector. Then $\boldsymbol{Z}$ is a Gaussian vector if and only if $\sum_{i=1}^n a_i Z_i$ has normal distribution for any $a_i \in \mathbb{R}$.*

▶ **Proposition 1.20.** *Suppose $\{\boldsymbol{X}_n\}_{n=1}^\infty$ is a sequence of Gaussian random vectors and $\lim_{n\to\infty} \boldsymbol{X}_n = \boldsymbol{X}$, a.s.. If*

$$\boldsymbol{\mu} := \lim_{n\to\infty} \mathbb{E}[\boldsymbol{X}_n], \quad K := \lim_{n\to\infty} \mathrm{Cov}(\boldsymbol{X}_n)$$

exists. Then, $\boldsymbol{X}$ is Gaussian with mean $\boldsymbol{\mu}$ and covariance matrix K.

2 The Brownian Motion

2.1 Stochastic Processes

▶ **Definition 2.1.** Let T be a set, $(E, \mathcal{E})$ be a measurable set. A *stochastic process* indexed by T taking its values on $(E, \mathcal{E})$ is a collection of measurable mappings $X = (X_t)_{t \in T}$ from a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to $(E, \mathcal{E})$. The space $(E, \mathcal{E})$ is called a *state space*.

Most of the time, the measurable space $(E, \mathcal{E}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, then $X = (X_t)_{t \in T}$ is a collection of random variables X_t. When $(E, \mathcal{E}) = (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, then the measurable mappings X_t are random vectors. The set T may be thought as time.

$T = \mathbb{R}_+$	continuous time stochastic process,
$T = \mathbb{N} = \{1, 2, \ldots, \}$	discrete time stochastic process.

▶ **Definition 2.2** (law of stochastic process)**.** The law of the stochastic process X is the probability measure

$$\mathbb{P}_X = \mathbb{P} \circ X^{-1} \quad \text{on } (\mathbb{R}^T, \mathcal{B}(\mathbb{R}^T)).$$

▶ **Definition 2.3** (version)**.** The processes X and X' defined on probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\Omega', \mathcal{F}', \mathbb{P}')$ having the same state space $(E, \mathcal{E})$ are called *versions* of each other if for any finite sequences $t_1, \ldots, t_n$ and sets $A_1, \ldots, A_n \in \mathcal{E}$,

$$\mathbb{P}(X_{t_1} \in A_1, \ldots, X_{t_n} \in A_n) = \mathbb{P}'(X'_{t_1} \in A_1, \ldots, X'_{t_n} \in A_n).$$

2.2 Definition of Brownian Motion

▶ **Definition 2.4** (Brownian motion)**.** A stochastic process $B = (B_t)_{t \geq 0}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called a *Brownian motion process* or a *Wiener process* if $B_0 = 0$ a.s. and

1. The mapping $t \mapsto B_t$ is continuous from $\mathbb{R}_+$ to $\mathbb{R}$.
2. B has *stationary increments*, i.e.
$$B_{t+r} - B_r \overset{\text{law}}{=} B_t \quad \text{for any } t, r \geq 0.$$
3. B has *independent increments*, i.e.
$$B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}}$$
are independent random variables for any choice of $0 \leq t_0 < t_1 < \cdots < t_n$ with $n \geq 1$.
4. $B_t \overset{\text{law}}{=} N(0, \sigma^2 t)$ for any $t > 0$, where $\sigma > 0$ is given and fixed constant.

▶ **Remark.** It can be shown using characteristic functions that the conditions 1, 2, 3 imply 4.

▶ **Remark.** If $\sigma^2 = 1$ in 4, then B is said to be a *standard Brownian motion*.

▶ **Theorem 2.5.** *Standard Brownian motion exists.*

▶ **Definition 2.6** (Gaussian process)**.** A stochastic process $Y = (Y_t)_{t \geq 0}$ is called a *Gaussian process* if for all $0 \leq t_1 < t_2 < \cdots < t_n$, the vector $(Y_{t_1}, Y_{t_2}, \ldots Y_{t_n})$ is a Gaussian vector.

Based on the Gaussian process, we have another characterization of a Brownian motion.

▶ **Theorem 2.7.** *A process $B = (B_t)_{t \geq 0}$ is a Brownian motion if and only if*

(a) *B is a Gaussian process.*

(b) *$t \to B_t$ is continuous.*

(c) $\mathbb{E}[B_t] = 0$.

(d) $\operatorname{Cov}(B_t, B_s) = \sigma^2 (t \wedge s)$*, for $t, s \geq 0$.*

▶ **Example 2.8.**

1. For $x \in \mathbb{R}$, the process $X_t^x = x + B_t$ is called the Brownian motion started at x.
2. More generally, the general Brownian motion process $X = (X_t^x)_{t\geq 0}$, defined as $X_t^x = \mu t + \sigma B_t$ is called *Brownian motion with drift started at x.* Using Theorem 1.13 one can obtains $X_t \sim N(x + \mu t, \sigma^2 t)$.

2.3 Properties of Brownian Motion

▶ **Theorem 2.9** (invariance properties of SBM)**.** *Let $B = (B_t)_{t\geq 0}$ be a standard Brownian motion defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Then each of the following processes is a standard Brownian motion as well enum*

(1) Renewal Property. *For a fixed $T > 0$,*

$$(B_{T+t} - B_T)_{t\geq 0}.$$

(2) Time-reversal. *For a fixed $T > 0$,*

$$(B_{T-t} - B_T)_{t\in[0,T]}.$$

(3) Reflection property.

$$(-B_t)_{t\geq 0}.$$

(4) Time-inversion.

$$(tB_{1/t})_{t\geq 0} = \begin{cases} tB_{1/t} & t \geq 0, \\ 0 & t = 0. \end{cases}$$

(5) Brownian scaling. *For fixed $\rho > 0$,*

$$\left(\frac{B_{\rho t}}{\sqrt{\rho}}\right)_{t\geq 0}.$$

▶ **Proposition 2.10** (law of large number of SBM)**.** *If $B = (B_t)_{t\geq 0}$ is a standard Brownian motion defined on $(\Omega, \mathcal{F}, \mathbb{P})$, then*

$$\lim_{t\to\infty} \frac{B_t}{t} = 0, \quad a.s..$$

▶ **Theorem 2.11** (the law of iterated logarithm for BM)**.** *If $B = (B_t)_{t\geq 0}$ is a SBM defined on $(\Omega, \mathcal{F}, \mathbb{P})$, then*

$$\limsup_{t\to\infty} \frac{B_t}{\sqrt{2t\log\log t}} = 1, \quad a.s.,$$
$$\liminf_{t\to\infty} \frac{B_t}{\sqrt{2t\log\log t}} = -1, \quad a.s.,$$
$$\limsup_{t\to 0} \frac{B_t}{\sqrt{2t\log\log(1/t)}} = 1, \quad a.s.,$$
$$\liminf_{t\to 0} \frac{B_t}{\sqrt{2t\log\log(1/t)}} = -1, \quad a.s..$$

▶ **Proposition 2.12** (SBM is nowhere differentiable)**.** *Let $B = (B_t)_{t\geq 0}$ be a SBM defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Then there exist $Z \in \mathcal{F}$ with $\mathbb{P}(Z) = 0$ such that for each $\omega \in \Omega \setminus Z$, the function $t \mapsto B_t(\omega)$ is nowhere differentiable on $\mathbb{R}_+$.*

3 Continuous Martingales

▶ **Definition 3.1.** A filtration on the measurable space $(\Omega, \mathcal{F})$ is an increasing family $(\mathcal{F}_t)_{t\geq 0}$ of sub-σ-algebras of $\mathcal{F}$. For each t, we have a σ-algebra $\mathcal{F}_t$ and $\mathcal{F}_s \subset \mathcal{F}_t$ for $s < t$. A measurable space $(\Omega, \mathcal{F})$ with a filtration $(\mathcal{F}_t)_{t\geq 0}$ is said to be a *filtered* space.

▶ **Definition 3.2.** A stochastic process $X = (X_t)_{t\geq 0}$ on $(\Omega, \mathcal{F})$ is *adapted* to the filtration $(\mathcal{F}_t)_{t\geq 0}$ if X_t is $\mathcal{F}_t$ measurable for each $t \geq 0$.

Then, any process X is adapted to its *natural filtration*

$$\mathcal{F}_t^X \coloneqq \sigma(X_s, s \leq t).$$

Consider an supplementary filtration

$$\mathcal{F}_{t+} = \bigcap_{s<t} \mathcal{F}_t.$$

▶ **Definition 3.3** (right-continuous filtration)**.** If $\mathcal{F}_t = \mathcal{F}_{t+}$, then the filtration is called right-continuous.

▶ **Definition 3.4** (martingale)**.** A stochastic process $X = (X_t)_{t\geq 0}$ is called a *martingale* with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$ if

(1) X is adapted to $(\mathcal{F}_t)_{t\geq 0}$.

(2) X_t is integrable for all $t \geq 0$, i.e. $\mathbb{E}[| X_t |] < \infty$.

(3) For all $0 \leq s \leq t$,
$$\mathbb{E}[X_t \mid \mathcal{F}_s] = X_s.$$

▶ **Definition 3.5** (stopping time)**.** A mapping $\tau : \Omega \to [0, \infty]$ is called a *stopping time* with respect to a filtration $(\mathcal{F}_t)_{t\geq 0}$ if for every $t \geq 0$,
$$\{\tau \leq t\} \in \mathcal{F}_t.$$

Based on the stopping time, we have the corresponding σ-algebra, $\mathcal{F}_\tau$,

$$\mathcal{F}_\tau \coloneqq \{A \in \mathcal{F} \mid A \cap \{\tau \leq t\} \in \mathcal{F}_t,\ \forall t \geq 0\}.$$

A stopping time may be thought of as the first time some physical event happens.

(1) Let $X = (X_t)_{t\geq 0}$ be a continuous process and let

$$D_A \coloneqq \inf\{t \geq 0 \mid X_t \in A\},$$

where A is a *closed set*. Then D_A is a stopping time with respect to the natural filtration $(\mathcal{F}_t^X)_{t\geq 0}$.

(2) Let $X = (X_t)_{t\geq 0}$ be a continuous process and let

$$T_A \coloneqq \inf\{t \geq 0 \mid X_t \in A\},$$

where A is an *open set*. Then T_A is a stopping time with respect to the filtration $(\mathcal{F}_{t+}^X)_{t\geq 0}$.

▶ **Theorem 3.6** (Doob's Optional Sampling Theorem)**.** *Let $X = (X_t)_{t\geq 0}$ be a continuous martingale. If X is uniformly integrable, then for any stopping time τ*

$$X_\tau = \mathbb{E}[X_\infty \mid \mathcal{F}_\tau].$$

In particular, $\mathbb{E}[| X_\tau |] < \infty$ and $\mathbb{E}[X_\tau] = \mathbb{E}[X_0]$. Moreover, if $\sigma < \tau$ a.s. is another stopping time, then

$$\mathbb{E}[X_\tau \mid X_\sigma] = X_\sigma \qquad a.s..$$

▶ **Theorem 3.7.** *Let X be a continuous martingale and τ be a stopping time with respect to $(\mathcal{F}_t)_{t\geq 0}$. Then the stopped process $X^\tau \coloneqq (X_{t\wedge\tau})_{t\geq 0}$ is a martingale with respect to the same filtration.*

▶ **Theorem 3.8** (Doob's maximal inequality). *Let $X = (X_t)_{t\geq 0}$ be a continuous martingale such that $\mathbb{E}[|X_t|^p] < \infty$, $t \geq 0$ for some $p \geq 1$. Then for every T and $\lambda > 0$, we have*

$$\mathbb{P}\left(\max_{0\leq t\leq T}|X_t| \geq \lambda\right) \leq \frac{\mathbb{E}[|X_T|^p]}{\lambda^p},$$

and if $p > 1$,

$$\mathbb{E}\left[\max_{0\leq t\leq T}|X_t|^p\right] \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}[|X_T|^p].$$

▶ **Theorem 3.9** (Doob's convergence theorem). *Let $X = (X_t)_{t\geq 0}$ be a uniformly integrable continuous martingale. Then there exist a limit,*

$$X(\omega) = \lim_{t\to\infty} X_t(\omega).$$

3.1 Martingales Consisting of Brownian Motions

▶ **Remark** (independence). For processes $(X_t)_{t\in I}$ and $(Y_t)_{t\in J}$, the corresponding σ-algebra $\sigma(X_t, t \in I)$ and $\sigma(Y_t, t \in J)$ are generated by the π-systems of sets

$$\{X_{t_1} \leq x_1, \ldots, X_{t_n} \leq x_n\}, \quad \{Y_{s_1} \leq y_1, \ldots, Y_{s_n} \leq y_m\}$$

where $t_1, \ldots, t_n \in I$, $s_1, \ldots, s_m \in J$, and $x_1, dots, x_n, y_1, \ldots, y_m \in \mathbb{R}$. Then the process X and Y are independent if and only if the random vectors

$$(X_{t_1} \leq x_1, \ldots, X_{t_n} \leq x_n), \quad (Y_{s_1} \leq y_1, \ldots, Y_{s_n} \leq y_m)$$

are independent for any $t_1, \ldots, t_n \in I$ and $s_1, \ldots, s_m \in J$.

▶ **Lemma 3.10** (Markov property of SBM). *Let $B = (B_t)_{t\geq 0}$ be a standard Brownian motion. Fix $T > 0$ and define $W = (W_t)_{t\geq 0}$, where $W_t \coloneqq B_{t+T} - B_T$. Then W is a standard Brownian motion and it is independent of $\mathcal{F}_T^B$. In particular $B_t - B_T$ is independent of $\mathcal{F}_T^B$ for all $t > T$.*

▶ **Example 3.11.**

(1) Let $B = (B_t)_{t\geq 0}$ be a standard one-dimensional Brownian motion. Then B is a martingale with respect to the natural filtration $\mathcal{F}_t^B$.

(2) The following processes constructed from one-dimensional Brownian motion are martingales

$$(B_t^2 - \sigma^2 t)_{t\geq 0}, \quad \left(\exp\left(\lambda B_t - \frac{\lambda^2\sigma^2}{2}t\right)\right)_{t\geq 0}, \qquad \lambda \in \mathbb{R}.$$

(3) For any convex function φ, provided $\mathbb{E}[|\varphi(B_t)|] < \infty$ for all $t \geq 0$, the process $(\varphi(B_t))_{t\geq 0}$ is a submartingale.

▶ **Theorem 3.12.** *Let $f(t, x)$ be a function on $\mathbb{R} \times \mathbb{R}^d$ which is continuously differentiable one in t and twice in x and satisfies the following estimate*

$$|f(t,x)| + \left|\frac{\partial f(t,x)}{\partial t}\right| + \sum_{j=1}^d \left|\frac{\partial f(t,x)}{\partial x_j}\right| + \sum_{j,k=1}^d \left|\frac{\partial^2 f(t,x)}{\partial x_j \partial x_k}\right| \leq K e^{K(t+|x|)}$$

for some $K > 0$. Let $B = (B_t)_{t\geq 0}$ be a standard d-dimensional Brownian motion started at a fixed point B_0. Then

$$M_t = f(t, B_t) - f(0, B_0) - \int_0^t Lf(r, B_r)\mathrm{d}r$$

is a martingale, where

$$Lf(t,x) = \frac{\partial f(t,x)}{\partial t} + \frac{1}{2}\Delta f(t,x).$$

3.2 Wald's Identities

The following result is an corollary from the optional stopping theorem applied on the Brownian motion.

▶ **Theorem 3.13** (Wald's identities)**.** *Let $B = (B_t)_{t\geq 0}$ be a Brownian motion started at x and let τ be a stopping time. Then*

$$\mathbb{E}[\tau] < \infty \quad \Longrightarrow \quad \mathbb{E}[B_\tau^2] < \infty, \quad \mathbb{E}[B_\tau] = x, \quad \mathbb{E}[B_\tau^2] - x^2 = \mathbb{E}[\tau].$$

The following corollary is an application of Wald's identities.

▶ **Corollary 3.14.** *Let $a < x < b$ and $B = (B_t)_{t\geq 0}$ be a standard Brownian motion started at x. Let*

$$\tau_a := \inf\{t \geq 0 : B_t = a\}, \quad \tau_b := \inf\{t \geq 0 : B_t = b\}, \quad \tau_{a,b} := \tau_a \wedge \tau_b.$$

Then, we have

$$\mathbb{P}(\tau_b < \tau_a) = \frac{x-a}{b-a}, \quad \mathbb{P}(\tau_a < \tau_b) = \frac{b-x}{b-a}, \quad \mathbb{E}[\tau_{a,b}] = \frac{a^2(b-x) + b^2(x-a)}{(b-a)} - x^2.$$

4 Markov Process

In this section, we will discuss the Markov processes, (strong) Markov property and prove that the Brownian motion is a strong Markov process.

4.1 Markov Property

We have shown that if we fixed time s, then the increment of the Brownian motion $(B(t) - B(s))_{t \geq s}$ does not depend on the process $(B_v)_{0 \leq v \leq s}$. That is

$$B(t) = \underbrace{(B(t) - B(s))}_{\perp \mathcal{F}_s^B} + \underbrace{B(s)}_{\mathcal{F}_s^B \text{ measurable}}$$

This property is called the *Markov property.*

► **Definition 4.1.** A stochastic process $X = (X_t)_{t \geq 0}$ on a state space $(E, \mathcal{E})$ is called a *Markov process* if

$$\mathbb{P}(X_t \in B \mid \mathcal{F}_s^X) = \mathbb{P}(X_t \in B \mid X_s), \qquad 0 \leq s \leq t, \quad \forall B \in \mathcal{E} \tag{MP1}$$

(i.e. B is $\mathcal{E}$ measurable). If $(\mathcal{F}_t)_{t \geq 0}$ is a filtration with $\mathcal{F}_t^X \subset \mathcal{F}_t$ for $t \geq 0$, then X is a Markov process with respect to $(\mathcal{F}_t)_{t \geq 0}$ if (MP1) holds with $\mathcal{F}_s^X$ replaced by $\mathcal{F}_s$.

This definition means that the future behavior of the Markov process, given the entire past, depends only on the present state of the process.

One can use standard arguments of conditional expectation shows that (MP1) is *equivalent* to one of the following conditions

$$\mathbb{E}[f(X_t) \mid \mathcal{F}_s^X] = \mathbb{E}[f(X_t) \mid X_s], \qquad 0 \leq s \leq t, \quad f \text{ is measurable and bounded.} \tag{MP2}$$

$$\mathbb{P}(B \mid \mathcal{F}_s) = \mathbb{P}(B \mid X_s), \qquad s \geq 0, \quad B \in \sigma(X_t,\ t \geq s). \tag{MP3}$$

$$\mathbb{E}[Y \mid \mathcal{F}_s] = \mathbb{E}[Y \mid X_s], \qquad Y \text{ is bounded and } \sigma(X_t,\ t \geq s) \text{ measurable.} \tag{MP4}$$

► **Definition 4.2** (transition probability)**.** When $(E, \mathcal{E})$ is "nice" measurable space (for example $\mathbb{R}^d$), then for any $t \geq s$, there exist a *version* $P_{s,t}(x, B)$ of $\mathbb{P}(X_t \in B \mid X_s = x)$ such that

1. For any $x \in E$, $B \to P_{s,t}(x, B)$ is a probability measure on $\mathcal{E}$.
2. For any $B \in \mathcal{E}$, $x \to P_{s,t}(x, B)$ is $\mathcal{E}$ measurable.

We will call $P_{s,t}(x, B)$ a set of *transition probabilities* for the process.

Using $P_{s,t}(x, B)$ we obtain

$$\mathbb{P}(X_t \in B \mid X_s) = P_{s,t}(X_s, B).$$

and hence (MP1) can be rewritten as follows

$$\mathbb{P}(X_t \in B \mid \mathcal{F}_s^X) = P_{s,t}(X_s, B). \tag{MP5}$$

► **Definition 4.3** (initial distribution)**.** A measure π defined by $\pi(B) = \mathbb{P}(X_0 \in B)$ is called the *initial distribution of the process.* If the initial distribution of a Markov process X is π, we will write $\mathbb{P}_\pi$. In case π is degenerate distribution at x, that is $\mathbb{P}(X_0 = x) = 1$, we will write $\mathbb{P}_x$.

The initial distribution and transition probabilities allows us to find a finite-dimensional distributions of the Markov process.

► **Proposition 4.4.** *Let $X = (X_t)_{t \geq 0}$ be a Markov process on a state space $(E, \mathcal{E})$ with transition probabilities $P_{s,t}$ and the initial distribution π. Then, for any $n \geq 1$, $B_0, \ldots, B_n \in \mathcal{E}$, and $0 = t_0 < t_1 \leq t_2 \leq \cdots \leq t_n$, we obtain the finite dimensional distribution of a Markov process X as follows*

$$\begin{aligned}\mathbb{P}(X_0 \in B_0, X_{t_1} \in B_1, \ldots, X_{t_n} \in B_n) \\ = \int_{B_0} \pi(\mathrm{d}x_0) \int_{B_1} P_{t_0,t_1}(x_0, \mathrm{d}x_1) \int_{B_2} P_{t_1,t_2}(x_1, \mathrm{d}x_2) \cdots \int_{B_n} P_{t_{n-1},t_n}(x_{n-1}, \mathrm{d}x_n).\end{aligned}$$

A special case of the Proposition 4.4 will be $n = 2$. Let $0 \leq s < t < u$ and B be a measurable set, we have

$$\begin{aligned} P_{s,u}(X_s, B) &= \mathbb{P}(X_u \in B \mid X_s) \quad \text{Definition} \\ &= \mathbb{P}(X_u \in B \mid \mathcal{F}_s^X) \quad \text{Markov Property} \\ &= \mathbb{E}[\mathbb{P}(X_u \in B \mid \mathcal{F}_t^X) \mid \mathcal{F}_s^X] \quad \text{Tower property of conditional probability} \\ &= \mathbb{E}[\mathbb{P}(X_u \in B \mid X_t) \mid X_s] \quad \text{Markov Property} \\ &= \mathbb{E}[P_{t,u}(X_t, B) \mid X_s] = \int_E P_{s,t}(X_s, \mathrm{d}y) P_{t,u}(y, B). \end{aligned}$$

Taking $X_s = x$, we have the *Chapman-Kolmorogov* equation

$$P_{s,u}(x, B) = \int_E P_{s,t}(x, \mathrm{d}y) P_{t,u}(y, B). \tag{4.1}$$

Given a Markov process, we can find its initial distribution and transition probabilities. However, *how we obtain a Markov process?* It is sufficient to obtain a Markov process by specifying the initial distribution and the set of transition probabilities and satisfies the C-K equation (4.1).

▶ **Definition 4.5.** The collection of functions

$$\{P_{s,t}(x, B),\ 0 \leq s < t < \infty,\ x \in E,\ B \in \mathcal{E}\}$$

is called a set of *Markov transition probabilities* if it satisfies the conditions in the Definition 4.2, and the C-K equation holds, i.e.

$$P_{s,u}(x, B) = \int_E P_{s,t}(x, \mathrm{d}y) P_{t,u}(y, B), \qquad 0 \leq s < t < u, \quad x \in E, \quad B \in \mathcal{E}.$$

Now given a family of Markov transition probabilities we can construct a Markov process with these transition probabilities.

▶ **Proposition 4.6.** *Given a set of Markov transition probabilities and initial distribution function π on $(E, \mathcal{E})$. One can construct a Markov process $X = (X_t)_{t \geq 0}$ on $(E, \mathcal{E})$ with these transition probabilities by specifying its finite-dimensional distributions as follows: for any $n \geq 1$, $B_0, \cdots, B_n \in \mathcal{E}$, and $t_1 \leq t_2 \leq \cdots \leq t_n$,*

$$\mathbb{P}(X_0 \in B_0, X_{t_1} \in B_1, \cdots X_{t_n} \in B_n) = \int_{B_0} \pi(\mathrm{d}x_0) \int_{B_1} P_{t_0,t_1}(x_0, \mathrm{d}x_1) \int_{B_2} P_{t_1,t_1}(x_1, \mathrm{d}x_2) \cdots \int_{B_n} P_{t_{n-1},t_n}(x_{n-1}, \mathrm{d}x_n).$$

4.2 Homogeneous Markov Process

The Markov transition probabilities are called homogeneous if for any $s \leq t$, $x \in E$, and $B \in \mathcal{E}$, we have

$$P_{s,t}(x, B) = P_{0,s-t}(x, B) \tag{4.2}$$

Then the Chapman-Kolmorogov equation (4.1) for homogeneous Markov transition probabilities can be written as

$$P_{s+t}(x, B) = \int_E P_s(x, \mathrm{d}y) P_t(y, B) \tag{4.3}$$

We can then define a family of operators $(P_t)_{t \geq 0}$, where each operator acts on a bounded measurable function f as follows

$$P_t f(x) = \int_E P_t(x, \mathrm{d}y) f(y) = \mathbb{E}_x[f(X_t)], \qquad x \in E \tag{4.4}$$

Intuitively, this operator represents: *where will be the underlying process $f(X)$ locate after time t if the process start at x.*

Then the Chapman-Kolmorogov equation (4.1) is equivalent as

$$\begin{aligned} P_{s+t}f(x) &= \int_E P_{s+t}(x, \mathrm{d}z)f(z) \quad \text{Definition of operator} \\ &= \int_E P_s(x, \mathrm{d}y) \int_E P_t(y, \mathrm{d}z)f(z) \quad \text{Using (4.3)} \\ &= \int_E P_s(x, \mathrm{d}y)P_t f(y) \\ &= P_s P_t f(x), \end{aligned}$$

which can be written as follows

$$P_{s+t} = P_s P_t. \tag{4.5}$$

If a Markov transition probability $P_t(x, B)$ can be represented as

$$P_t(x, B) = \int_B p_t(x, y)\ \mathrm{d}y$$

for some function $p_t(x, y)$ and for all $t > 0$, $x \in \mathbb{R}$ and $B \in \mathcal{B}(\mathbb{R})$, then $p_t(x, y)$ is called the *transition density*.

4.3 Examples of Markov Process

▶ **Theorem 4.7** (SBM is a Markov process)**.** *Let $B = (B_t)_{t\geq 0}$ be a standard Brownian motion defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Then B is a Markov process with respect to its natural filtration $(\mathcal{F}_t^B)_{t\geq 0}$.*

Proof. Let f be a bounded measurable function and fix $0 \leq s < t$. We will verify (MP2). Note that

$$\mathbb{E}[f(B_t) \mid \mathcal{F}_s^B] = \mathbb{E}[f(B_t - B_s + B_s) \mid \mathcal{F}_s^B].$$

By the Markov property of the Brownian motion (Lemma 3.10), $B_t - B_s$ is independent of $\mathcal{F}_s^B$. Hence we can apply independent Lemma 1.12 by treating $Y = B_t - B_s$, $X = B_s$ and $\Phi(x, y) = f(x + y)$, and we have

$$\begin{aligned} \mathbb{E}[f(B_t) \mid \mathcal{F}_s^B] &= \mathbb{E}[\Phi(B_t - B_s, B_s) \mid \mathcal{F}_s^B] \\ &= \mathbb{E}[\Phi(B_t - B_s, B_s) \mid B_s] \\ &= \mathbb{E}[f(B_t) \mid B_s]. \end{aligned}$$

□

▶ **Example 4.8** (SBM is homogeneous Markov process)**.** It follows from Theorem 4.7 that one can take

$$P_t(x, A) = \mathbb{P}(x + B_t \in A) = \int_{y:\ x+y\in A} \frac{1}{\sqrt{2\pi t}} \mathrm{e}^{-y^2/(2t)}\ \mathrm{d}y = \int_B p_t(x, y)\ \mathrm{d}y$$

as transition probabilities. Here

$$p_t(x, y) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(x - y)^2}{2t}\right), \qquad x, y \in \mathbb{R},\ t > 0 \tag{4.6}$$

is the transition density of the standard Brownian motion.

It is clear that it satisfies the conditions in the Definition 4.2. Moreover, from Theorem 4.7,

$$\mathbb{P}(B_t \in A \mid B_s) = \mathbb{P}(x + B_t - B_s \in A \mid B_s = x)|_{x=B_s} = P_{t-s}(B_s, A).$$

Remain to check the Chapman-Kolmorogov equation holds,

$$P_t f(x) = \int_{-\infty}^{\infty} P_t(x, \mathrm{d}y)f(y) = \int_{-\infty}^{\infty} \mathbb{P}(x + B_t \in \mathrm{d}y)f(y) = \mathbb{E}[f(x + B_t)].$$

Then,

$$\begin{aligned}P_s(P_tf(x)) &= P_s\left(\mathbb{E}[f(x+B_t)]\right)\\ &= P_s\left(\mathbb{E}[f(x+B_{t+s}-B_s)]\right)\\ &= \mathbb{E}[f(x+B_{t+s}+B_s-B_s)]\\ &= \mathbb{E}[f(x+B_{t+s})] = P_{t+s}f(x).\end{aligned}$$

Hence, the transition probabilities satisfies 1) Definition 4.2, 2) homogeneous condition (4.2), 3) Chapman-Kolmorogov equation. We can conclude that, B is a homogeneous Markov process with transition density given by (4.6).

▶ **Example 4.9** (examples of Markov process)**.** The following processes are Markov processes,

1. Brownian motion with a drift $(\mu t + \sigma B_t)_{t\geq 0}$, where $B = (B_t)_{t\geq 0}$ is a Brownian motion. It is a continuous Markov process.

2. Ornstein-Uhlenbeck processes, both stationary $(V_t)_{t\geq 0}$, where

$$V_t = \frac{\sigma}{\sqrt{2\beta}}\mathrm{e}^{-\beta t}B_{\mathrm{e}^{2\beta t}}, \qquad t\geq 0,$$

   and non-stationary $(V_t)^{(v)}_{t\geq 0}$, where

$$V_t^{(v)} = v\mathrm{e}^{-\beta t} + \frac{\sigma}{\sqrt{2\beta}}\mathrm{e}^{-\beta t}B_{\mathrm{e}^{2\beta t}-1}.$$

   Here $B = (B_t)_{t\geq 0}$ is a Brownian motion. V_t and $V_t^{(v)}$ have the same transition probabilities, but different initial distributions. Ornstein-Uhlenbeck process is continuous.

3. Geometric Brownian motion $S = (S_t)_{t\geq 0}$ defined as

$$S_t = S_0\exp(\mu t + \sigma B_t), \qquad t\geq 0,$$

   where $B = (B_t)_{t\geq 0}$ is a standard Brownian motion. Geometric Brownian motion is a continuous Markov process.

4. Generally, If $X = (X_t)_{t\geq 0}$ be a Markov process and f is a bijection measurable function. Then $Y = (Y_t)_{t\geq 0} \coloneqq (f(X_t))_{t\geq 0}$ is a Markov process as well.

4.4 Strong Markov Property of Brownian Motion

Markov property (Lemma 3.10) ensures that for any fixed time T, the process $(B_{t+T} - B_t)_{t\geq 0}$ is a Brownian motion and independent of $\mathcal{F}_T^B$. Now we will show that it is possible to replace the fixed time T with a stopping time τ.

▶ **Lemma 4.10** (discrete to continuous stopping times)**.** *Given a stopping time $\tau : \Omega \to [0,\infty]$ with respect to a filtration $(\mathcal{F}_t)_{t\geq 0}$ define*

$$\tau_n = \begin{cases} j2^{-n} & \text{if } (j-1)2^{-n} \leq \tau < j2^{-n}, \qquad j = 1,2,\dots,\\ \infty & \text{if } \tau = \infty\end{cases}$$

for $n\geq 1$. Then

(a) *Each τ_n is a stopping time with respect to $(\mathcal{F}_{t+})_{t\geq 0}$.*

(b) *$\tau_n \downarrow \tau$ as $n\to\infty$.*

(c) *$\mathcal{F}_{\tau+} = \cap_{h>0}\mathcal{F}_{\tau+h}$.*

▶ **Theorem 4.11** (strong Markov property of SBM)**.** *Let $B = (B_t)_{t\geq 0}$ be a standard Brownian motion and $\tau < \infty$ a.s.. Then $W = (W_t)_{t\geq 0}$, where $W_t = B_{\tau+t} - B_\tau$ is a standard Brownian motion as well. Moreover, W is independent of $\mathcal{F}^B_{\tau+}$.*

Proof. We would like to prove the independent using characteristic functions. Let τ_n be a sequence of stopping times constructed from τ in Lemma 4.10. For any $0 \le s < t$, $\theta \in \mathbb{R}$ and $C \in \mathcal{F}_{\tau+}$, we have

$$\begin{aligned}
\mathbb{E}[\mathrm{e}^{\mathrm{i}\theta(B_{\tau_n+t}-B_{\tau_n+s})}\mathbf{1}_C] &= \sum_{j=1}^{\infty}\mathbb{E}[\mathrm{e}^{\mathrm{i}\theta(B_{j2^{-n}+t}-B_{j2^{-n}+s})}\mathbf{1}_{\tau_n=j2^{-n}}\mathbf{1}_C] \\
&= \sum_{j=1}^{\infty}\mathbb{E}\left[\mathrm{e}^{\mathrm{i}\theta(B_{j2^{-n}+t}-B_{j2^{-n}+s})}\underbrace{\mathbf{1}_{(j-1)2^{-n}\le\tau<j2^{-n}}\mathbf{1}_C}_{\mathcal{F}_{j2^{-n}}\text{ measurable}}\right] \\
\text{(Markov property. No randomness)} &= \sum_{j=1}^{\infty}\mathbb{E}[\mathrm{e}^{\mathrm{i}\theta(B_{j2^{-n}+t}-B_{j2^{-n}+s})}]\mathbb{P}(\{\tau_n = j2^{-n}\}\cap C) \\
&= \mathbb{E}[\mathrm{e}^{\mathrm{i}\theta B_{t-s}}]\sum_{j=1}^{\infty}\mathbb{P}(\{\tau_n = j2^{-n}\}\cap C) \\
&= \mathbb{E}[\mathrm{e}^{\mathrm{i}\theta B_{t-s}}]P(C).
\end{aligned}$$

Now letting $n \to \infty$ and using the continuity of Brownian motion with the DCT, we obtain

$$\mathbb{E}[\mathrm{e}^{\mathrm{i}\theta(B_{\tau+t}-B_{\tau+s})}\mathbf{1}_C] = \mathbb{E}[\mathrm{e}^{\mathrm{i}\theta B_{t-s}}]\mathbb{P}(C).$$

Similar calculation show that the same statement is true for finitely many increments. Namely, for $0 = t_0 < t_1 < \cdots < t_n$ and $\theta, \ldots, \theta_n \in \mathbb{R}$,

$$\mathbb{E}[\mathrm{e}^{\mathrm{i}\sum_{j=1}^{n}\theta_j(B_{\tau+t_j}-B_{\tau+t_{j-1}})}\mathbf{1}_C] = \prod_{j=1}^{n}\mathbb{E}[\mathrm{e}^{\mathrm{i}\theta_j B_{t_j-t_{j-1}}}\mathbb{P}(C)].$$

This shows that the increments of W are *independent* of each other and of $\mathcal{F}_{\tau+}$. Also, this shows that the increments of W have the *same distribution* as the increment of B. Finally, the continuity of W follows from that of B. We can then conclude that W is a Brownian motion and the process W is independent of $\mathcal{F}_{\tau+}$. □

4.5 Reflection Principal

▶ **Theorem 4.12** (reflection principal of SBM). *Fix $a \in \mathbb{R}$. Let $B = (B_t)_{t\ge 0}$ be a SBM and*

$$\tau_a = \inf\{t > 0 \mid B_t = a\}$$

be the first time which the Brownian motion B hits a. Then the process $\widetilde{B} = (\widetilde{B}_t)_{t\ge 0}$

$$\widetilde{B}_t = \begin{cases} B_t, & t < \tau_a, \\ 2a - B_t, & t \ge \tau_a \end{cases}$$

is a standard Brownian motion.

Proof. Consider the stochastic process $Z = (Z_t)_{t\ge 0}$ and $Y = (Y_t)_{t=0}^{\tau_a}$ where

$$Y_t \coloneqq B_t, \qquad 0 \le t \le \tau_a, \qquad Z_t \coloneqq B_{t+\tau_a} - B_{\tau_a} = B_{t+\tau_a} - a.$$

Since τ_a is a stopping time, therefore Z_t is a SBM independent of $\mathcal{F}_{\tau_a+}$, in particular, independent of Y. By the reflection property, $-Z$ is also a SBM independent of Y. Hence $(Y, Z) \overset{\text{law}}{=} (Y, -Z)$. The map

$$\phi(Y, Z) \mapsto (Y_t I_{t\le\tau_a} + (a + Z_{t-\tau_a})I(t > \tau_a))_{t\ge 0}$$

produces a continuous process, which have the same law as $\phi(Y, -Z)$. But $\phi(Y, Z) = B$ and $\phi(Y, -Z) = \widetilde{B}$. □

▶ **Corollary 4.13.** *Let $B = (B_t)_{t\geq 0}$ be a SBM. Let $S_t := \sup_{s\leq t} B_s$ be the running supremum of B. Then for all $a > 0$ and $t > 0$,*

$$\mathbb{P}(S_t \geq a) = \mathbb{P}(\tau_a \leq t) = 2\mathbb{P}(B_t \geq a) = \mathbb{P}(|\, B_t \,| \geq a).$$

Proof. Notice that

$$\begin{aligned}\mathbb{P}(S_t \geq a) &= \mathbb{P}(S_t \geq a, B_t > a) + \mathbb{P}(S_t \geq a, B_t \leq a) \\ &= \mathbb{P}(B_t > a) + \mathbb{P}(S_t \geq a, B_t \leq a).\end{aligned}$$

Let $\widetilde{B}$ be the process defined in Theorem 4.12, and put $\widetilde{S}_t = \sup_{s\leq t} \widetilde{B}_s$. Since B and $\widetilde{B}$ have the same distribution, one can see

$$\mathbb{P}(S_t \geq a, B_t \leq a) = \mathbb{P}(S_t \geq a, \widetilde{B}_t \leq a).$$

By definition of $\widetilde{B}$, on the event $\{t < \tau_a\} = \{S_t < a\}$ (B_t hasn't meet a), the trajectories of B and $\widetilde{B}$ coincide. Hence,

$$\begin{aligned}\mathbb{P}(S_t \geq a, B_t \leq a) &= \mathbb{P}(S_t \geq a, \widetilde{B}_t \leq a) \\ &= \mathbb{P}(\tau_a \leq t, 2a - B_t \leq a) \\ &= \mathbb{P}(S_t \geq a, B_t \geq a) = \mathbb{P}(B_t \geq a).\end{aligned}$$

Then

$$\mathbb{P}(\tau_a \leq t) = \mathbb{P}(S_t \geq a) = \mathbb{P}(B_t > a) + \mathbb{P}(B_t \geq a) = 2\mathbb{P}(B_t \geq a) = \mathbb{P}(|\, B_t \,| \geq a),$$

where the last equality follows from the symmetry of the SBM. □

4.6 Strong Markov Processes and Feller Processes

The following definition extend the Definition 4.1 to stopping time τ.

▶ **Definition 4.14** (strong Markov process)**.** Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$ be a filtered probability space. An adapted stochastic process $X = (X_t)_{t\geq 0}$ on the state space $(E, \mathcal{E})$ is called a *strong Markov process* with respect to $(\mathcal{F}_t)_{t\geq 0}$ if

$$\mathbb{P}(X_{\tau+s} \in B \mid \mathcal{F}_\tau) = \mathbb{P}(X_{\tau+s} \in B \mid X_\tau) = P_s(X_\tau, B), \tag{4.7}$$

on $\{\tau < \infty\}$ for transition probability $P_t(x, B)$, all stopping times τ, $s \geq 0$ and $B \in \mathcal{E}$.

▶ **Remark.** The state space E that we will consider will usually be either $\mathbb{R}^d$ or some nice subset of $\mathbb{R}^d$.

The meaning of the strong Markov property is similar to the Markov property. The behavior of the Markov process after stopping time depends only on the position of the process at the stopping time and does not depend on the behavior of the process before stopping time.

Recall that for a measurable function f and transition probabilities $P_t(x, B)$ of a *time homogeneous Markov process*,

$$P_t f(x) = \int_E P_t(x, \mathrm{d}y) f(y) = \mathbb{E}[f(X_t) \mid X_0 = x] = \mathbb{E}_x[f(X_t)].$$

Denote by $C_b(E)$ the set of continuous and bounded functions on E. We can further show that a process $X = (X_t)_{t\geq 0}$ is a strong Markov process if and only if for all $f \in C_b(E)$,

$$\mathbb{E}[f(X_{\tau+s}) \mid \mathcal{F}_\tau] = P_s f(X_\tau). \qquad \text{see (MP2)}. \tag{4.8}$$

▶ **Definition 4.15** (Feller property)**.** A Markov process $X = (X_t)_{t\geq 0}$ with transition probabilities P_t is said to have the *Feller property* if for any $f \in C_b(E)$, and $t \geq 0$, the function $P_t f \in C_b(E)$. In this case, we call X a *Feller process*.

▶ **Theorem 4.16.** *Let $X = (X_t)_{t\geq 0}$ be a right-continuous Feller process defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Then X is a strong Markov process with respect to the right-continuous filtration $(\mathcal{F}^X_{t+})_{t\geq 0}$.* ▶ Markov process + Feller property = Strong Markov Process

Proof. We need to verify (4.7). Let τ be an arbitrary stopping time, $s > 0$ and $f \in C_b(E)$. It is sufficient to show (4.8). By the definition of conditional expectations, it is sufficient to show that

$$\mathbb{E}[f(X_{\tau+s})\mathbf{1}_C] = \mathbb{E}[P_s f(X_\tau \mathbf{1}_C)], \qquad \forall C \in \mathcal{F}_{\tau+}. \tag{4.9}$$

Recall the Lemma 4.10, there exist a decreasing sequence of stopping times τ_n for $n \geq 1$ such that $\tau_n \downarrow \tau$. Then,

$$\begin{aligned}
\mathbb{E}[f(X_{\tau_n})\mathbf{1}_C] &= \sum_{j=1}^{\infty} \mathbb{E}[f(X_{j2^{-n}+s}) \underbrace{\mathbf{1}_{\tau_n = j2^{-n}}\mathbf{1}_C}_{\mathcal{F}_{j2^{-n}} \text{ measurable}}] \\
\text{(conditional mean formula)} &= \sum_{j=1}^{\infty} \mathbb{E}[\mathbf{1}_{\tau_n = j2^{-n}}\mathbf{1}_C\, \mathbb{E}[f(X_{j2^{-n}+s}) \mid \mathcal{F}_{j2^{-n}}]] \\
\text{(Markov property)} &= \sum_{j=1}^{\infty} \mathbb{E}[\mathbf{1}_{\tau_n = j2^{-n}}\mathbf{1}_C P_s f(X_{j2^{-n}})] \\
&= \mathbb{E}[P_s f(X_{\tau_n})\mathbf{1}_C].
\end{aligned}$$

As $n \to \infty$, since $\tau_n \downarrow \tau$, process X is right-continuous and f is continuous, therefore $f(X_{\tau_n+s})\mathbf{1}_C \to f(X_{\tau+s})\mathbf{1}_C$. Similarly argument can be applied by noticing $P_s f$ is continuous by Feller property, hence $P_s f(X_{\tau_n})\mathbf{1}_C \to P_s f(X_\tau)\mathbf{1}_C$. Therefore, since f and $P_s f$ are both bounded, hence we use DCT to obtain (4.9). □

► **Corollary 4.17** (SBM is strong Markov Process)**.** *Let $B = (B_t)_{t\geq 0}$ be a SBM. Then B is a strong Markov process with respect to the filtration $\mathcal{F}^B_{t+}$.*

Proof. SBM is a Markov process, therefore, it is sufficient o show that SBM satisfies the Feller property and apply Theorem 4.16. For a fixed $f \in C_b(\mathbb{R})$, we need to show that $P_t f \in C_b(\mathbb{R})$ for any fixed t. To prove that $P_t f(x)$ is continuous at x, note that

$$f(y + B_t) \to f(x + B_t),$$

as $y \to x$. Also $\sup_{y\in\mathbb{R}} |f(y)| < K < \infty$ for some K. Hence the DCT implies that as $y \to x$,

$$P_t f(y) = \mathbb{E}[f(y + B_t)] \to \mathbb{E}[f(x + B_t)] = P_t f(x),$$

and since x is arbitrary, therefore $P_t f \in C_b(\mathbb{R})$. □

► **Definition 4.18** (shift operator)**.** Define a family of measurable transformations θ_t, $t \geq 0$, by

$$X_s(\theta_t(\omega)) = X_{t+s}(\omega).$$

The operators θ_t are called the *shift operator*. The shift operator will do nothing but proceed the time with t unit. Clearly, $\theta_t \circ \theta_s = \theta_{t+s}$ and θ_t is measurable with respect to $\sigma(X_s, s \geq t)$ and $\sigma(X_s, s \geq 0)$.

Shift operators allow us to reformulate the Markov property in the following way. Let $\mathcal{F}^X_\infty = \sigma(\cup_{t\geq 0}\mathcal{F}^X_t)$. Then $X = (X_t)_{t\geq 0}$ is a Markov process *if and only if* for any $\mathcal{F}_\infty$ measurable bounded Z such that

$$\mathbb{E}_x[Z \circ \theta_t \mid \mathcal{F}^X_t] = \mathbb{E}_{X_t}[Z], \quad \text{a.s..} \tag{4.10}$$

More detail see Theorem 2.10 in link.

For a stopping time τ, we can further define a map $\theta_\tau : \Omega \mapsto \Omega$, by

$$\theta_\tau(\omega) = \theta_t(\omega), \qquad \text{if } \tau(\omega) = t.$$

Then the strong Markov property can also be reformulated as: the process $X = (X_t)_{t\geq 0}$ is a strong Markov process if and only if for any bounded $\mathcal{F}_\infty$ measurable Z,

$$\mathbb{E}_x[Z \circ \theta_\tau \mid \mathcal{F}_\tau] = \mathbb{E}_{X_\tau}[Z],$$

on $\{\tau < \infty\}$ for every stopping time τ, every $\mathcal{F}_\infty$ measurable Z and all x.

5 One-dimensional Regular Diffusion

5.1 Regularity

Let $(\Omega, \mathcal{F}, \mathbb{P}_x)$ be a probability space and let $X = (X_t)_{t\geq 0}$ be a *continuous strong Markov process* with respect to the filtration $\mathcal{F}^X_{t+}$ with values in the measurable space $(I, \mathcal{B}(I))$, such that $\mathbb{P}_x(X_0 = x) = 1$ for some $x \in I$. We will assume that the state space I is an interval in $\mathbb{R}$, which may be closed, open or semi-open, bounded or unbounded. We set $l = \inf I \geq -\infty$, and $r = \sup I \leq +\infty$.

We will start with *description of the behavior of X at an arbitrary point x*. The key result is the zero-one law.

▶ **Lemma 5.1** (Blumenthal 0-1 law). *The σ-algebra $\mathcal{F}^X_{0+}$ is $\mathbb{P}_x$ trivial. Namely, $\mathbb{P}_x(A) \in \{0, 1\}$ for any $A \in \mathcal{F}^X_{0+}$.*

Proof. Let $A \in \mathcal{F}^X_{0+}$ and $B \in \mathcal{F}^X_{\infty}$. Then

$$\begin{aligned} \mathbb{P}_x(A \cup B) &= \mathbb{E}_x[\mathbf{1}_A \cdot \mathbf{1}_B] \\ &= \mathbb{E}_x[\mathbf{1}_A\, \mathbb{E}_x[\mathbf{1}_B \mid \mathcal{F}^X_{0+}]] \\ \text{(Markov property)} &= \mathbb{E}_x[\mathbf{1}_A\, \mathbb{E}_{X_0}[\mathbf{1}_B]] \\ &= \mathbb{E}_x[\mathbf{1}_A \mathbb{P}_x(B)] \\ &= \mathbb{P}_x(A)\mathbb{P}_x(B). \end{aligned}$$

Taking $B = A$, we have $\mathbb{P}_x(A) = (\mathbb{P}_x(A))^2$. Hence either $\mathbb{P}_x(A) = 0$ or $\mathbb{P}_x(A) = 1$. ☐

For $x \in \mathbb{R}$, let

$$\overline{\tau}_x = \inf\{t > 0 \mid X_t \neq x\} \tag{5.1}$$

be the first time X leaves x. Note that $\overline{\tau}_x$ is the first hitting time of X to the open set $(-\infty, x) \cup (x, +\infty)$, hence $\overline{\tau}_x \in [0, \infty]$ is a stopping time with respect to $\mathcal{F}^X_{t+}$.

By continuity of X,

$$\{\overline{\tau}_x = 0\} = \bigcap_{n=1}^{\infty} \underbrace{\{X_s \neq x \text{ for some } s \in [0, 1/n]\}}_{\in \mathcal{F}^X_{1/n}} \in \mathcal{F}^X_{0+}.$$

Hence $\mathbb{P}_x(\overline{\tau}_x = 0) = \{0, 1\}$ by Lemma 5.1, and therefore

$$\mathbb{P}_x(\overline{\tau}_x > 0) = \{0, 1\}.$$

Characterization of $\mathbb{P}_x(\overline{\tau}_x > 0)$ and related probabilities.

(1) $\mathbb{P}_x(\overline{\tau}_x > 0)$ can be viewed as the *probability that we stay at the current position now.*

(2) $\mathbb{P}_x(\overline{\tau}_x = 0)$ can be viewed as the *probability that we exit the current position now.*

When $\mathbb{P}_x(\overline{\tau}_x > 0) = 0$, we say that x is *instantaneous*. When $\mathbb{P}_x(\overline{\tau}_x > 0) = 1$, we say that x is absorbing. Moreover, define

$$\overline{\tau}^+_x = \inf\{t > 0 : X_t > x\}, \qquad \overline{\tau}^-_x = \inf\{t > 0 : X_t < x\},$$

be respectively the first time X exceed x or become smaller than x. Based on this definition, we can further classify the instantaneous points into the following categories,

	$\mathbb{P}_x(\overline{\tau}^-_x = 0) = 1$	$\mathbb{P}(\overline{\tau}^-_x = 0) = 0$
$\mathbb{P}_x(\overline{\tau}^+_x = 0) = 1$	Instantaneous **Regular point**	Instantaneous **Lower boundary point**
$\mathbb{P}_x(\overline{\tau}^+_x = 0) = 0$	Instantaneous **Upper boundary point**	Absorbing

Next theorem will characterize the behavior of X at non-regular points, i.e. upper or lower boundary point and the absorbing point.

▶ **Lemma 5.2.** *Let $x \in I$,*

(1) *Let x is absorbing. If X started at x (or visit x), it stays forever. Namely, $\mathbb{P}_x(\overline{\tau}_x = \infty) = 1$.*

(2) *If x is a lower (upper) boundary point, then X never goes below (above) x. Namely, $\mathbb{P}_x(\overline{\tau}_x^- = \infty) = 1$, $(\mathbb{P}_x(\overline{\tau}_x^+ = \infty) = 1)$.*

Proof. We will prove (1), and (2) can be proved similarly.

$$\begin{aligned}
\mathbb{P}_x(\overline{\tau}_x < \infty) &= \mathbb{P}_x(\overline{\tau}_x < \infty, \overline{\tau}_x \circ \theta_{\overline{\tau}_x} = 0)^{1} \\
&= \mathbb{E}_x[\mathbf{1}_{\overline{\tau}_x < \infty} \mathbf{1}_{\overline{\tau}_x \circ \theta_{\overline{\tau}_x} = 0}] \\
&= \mathbb{E}_x\left[\mathbb{E}_x[\mathbf{1}_{\overline{\tau}_x < \infty} \mathbf{1}_{\overline{\tau}_x \circ \theta_{\overline{\tau}_x} = 0} \mid \mathcal{F}_{\overline{\tau}_x +}]\right] \quad \text{(Tower property)} \\
&= \mathbb{E}_x\left[\mathbb{E}_x[\mathbf{1}_{\overline{\tau}_x = 0} \circ \theta_{\overline{\tau}_x} \mid \mathcal{F}_{\overline{\tau}_x +}]\right] \quad (\mathbf{1}_{\overline{\tau}_x < \infty} \text{ is } \mathcal{F}_{\overline{\tau}_x +} \text{ measurable}) \\
&= \mathbb{E}_x\left[\mathbf{1}_{\overline{\tau}_x < \infty} \mathbb{E}_{X_{\overline{\tau}_x}}[\mathbf{1}_{\overline{\tau}_x = 0}]\right] \quad \text{(Strong markov property (4.10))} \\
&= \mathbb{E}_x[\mathbf{1}_{\overline{\tau}_x < \infty} \mathbb{P}_x(\overline{\tau}_x = 0)] \quad \text{(continuity of } X_t\text{)} \blacktriangleright \text{ why continuity?} \\
&= \mathbb{P}_x(\overline{\tau}_x < \infty)\mathbb{P}_x(\overline{\tau}_x = 0) \\
&= 0 \quad \text{by assumption that } x \text{ is absorbing.}
\end{aligned}$$

Hence we have proved (1). □

Hence we have the following classification for arbitrary point $x \in I$.

> For any $x \in I$, it is either *absorbing* or *instantaneous*. If x is instantaneous, then it is either an *upper boundary* or *lower boundary* or *regular*.

▶ **Example 5.3.** The following examples illustrate the above classification.

($E1$) Consider a SBM $B = (B_t)_{t \geq 0}$. This is a strong Markov process on $I = (-\infty, +\infty)$. *All points $x \in \mathbb{R}$ are regular.*

Proof. Let's fix $x \in \mathbb{R}$, then

$$\begin{aligned}
\mathbb{P}_x(\overline{\tau}_x^+ = 0) &= \mathbb{P}\left(\bigcup_{n \geq 1} (x + B_s) > x, \text{ for some } s \leq \frac{1}{n}\right) \\
&= \lim_{n \to \infty} \mathbb{P}\left((x + B_s) > x \text{ for some } s \leq \frac{1}{n}\right) \\
&\geq \lim_{n \to \infty} \mathbb{P}(B_{1/n} > 0) \qquad \blacktriangleright \text{ why? }^{2} = \frac{1}{2}.
\end{aligned}$$

By Blumenthal zero-one law, $\mathbb{P}_x(\overline{\tau}_x^+ = 0) = \{0, 1\}$, we obtain $\mathbb{P}_x(\overline{\tau}_x^+ = 0) = 1$. By symmetry $\mathbb{P}_x(\overline{\tau}_x^- = 0) = 1$, and hence $x \in \mathbb{R}$ is regular. □

($E2$) Let $B = (B_t)_{t \geq 0}$ be a SBM. Consider the *reflected Brownian motion* $X = (X_t)_{t \geq 0}$ given by $X_t = |\, B_t \,|$. This is a strong Markov process $I = [0, +\infty)$. *For $x \in (0, +\infty)$, x is regular. The boundary is a lower boundary point.*

Proof. Indeed, $\mathbb{P}_0(\overline{\tau}_0^- = 0) = 0$ since $|\, B_t \,| \geq 0$ for any $t \geq 0$. However, for the counter part,

$$\mathbb{P}_x(\overline{\tau}_x^+ = 0) = \lim_{n \to \infty} \mathbb{P}\left(x + B_s \neq x \text{ for some } s \leq \frac{1}{n}\right) \geq \lim_{n \to \infty} (B_{1/n} \neq 0) = 1 \tag{5.2}$$

[1] If the event $\overline{\tau}_x < \infty$ happened, this implies that there exist a finite time $t = \overline{\tau}_x$ such that $X_t \neq x$, then we shift the whole axis by $\overline{\tau}_x$, then clearly, it becomes we leave x at $t = 0$, namely $\overline{\tau}_x \circ \theta_{\overline{\tau}_x} = 0$, since $\overline{\tau}_x$ exists and finite.

[2] My guess will be: the event $\{B_s > 0\}$ for some $s \leq 1/n$ is much more general than $\{B_{1/n} > 0\}$. In fact, for the former case, we only need to find one $s \in [0, 1/n]$, but the latter case require specifically at $1/n$.

since for all $x \in \mathbb{R}$, it is a regular point for B_t. [3] Hence back to X_t, x is not absorbing, and therefore it is lower boundary. □

($E3$) Let $B = (B_t)_{t\geq 0}$ be a SBM started at x. Consider the *Brownian motion absorbed at the boundary* $X = (X_t)_{t\geq 0}$ given by $X_t = |\, B_{t\wedge\widetilde{\tau}_0}\,|$, where

$$\widetilde{\tau}_0 := \inf\{t \geq 0 : B_t = 0\}.$$

One can show that this is a strong Markov process on $[0, \infty)$. *For all $x > 0$, x is regular, and $x = 0$ is an absorbing point.*

($E4$) Similarly, we can define $X = (X_t)_{t\geq 0}$ absorbed at the boundaries of the interval $[-1, 1]$ by $X_t = |\, B_{t\wedge\tau_{-1,1}}\,|$, where

$$\tau_{-1,1} =:= \inf\{t \geq 0 : B_t \in \{-1, 1\}\}.$$

This is a strong Markov process on $I = [-1, 1]$. *The interval point $x \in (-1, 1)$ are regular while the boundary points $x \in \{-1, 1\}$ are absorbing.*

Main assumptions about $X = (X_t)_{t\geq 0}$

Denote the *hitting time* of X by

$$\tau_y := \inf\{t > 0 : X_t = y\},$$

where $\inf\{\emptyset\} = \infty$. Clearly, since $t \to X_t$ is continuous, then $X_{\tau_y} = y$ on $\{\tau_y < \infty\}$. We now list three main assumptions:

($R1$) The paths of X are continuous.

($R2$) X is a strong Markov process with respect to $\mathcal{F}^X_{t+}$ in the sense (4.10).

($R3$) X is *regular* in the sense that

$$\mathbb{P}_x(\tau_y < \infty) > 0$$

for any $x \in \mathrm{int}(I) = (\ell, r)$ and any $y \in I$.

► **Definition 5.4.** The process $X = (X_t)_{t\geq 0}$ satisfying assumptions ($R1$), ($R2$) and ($R3$) is called a *regular one-dimensional diffusion.*

Examples of regular diffusion are Brownian motion (with or without a drift), reflected Brownian motion and Ornstein-Uhlenbeck process. The points of the regular diffusion are classified as follows.

► **Lemma 5.5.** *Let $X = (X_t)_{t\geq 0}$ be a regular one-dimensional diffusion with a state space I. Then*

(1) *The right (left) endpoint of I is either*

 (P1) *not in I, or*

 (P2) *in I and absorbing, or*

 (P3) *in I and is an upper (lower) boundary point.*

(2) *Each point $x \in \mathrm{int}(I)$ is regular.*

(3) *Conversely, suppose that X satisfies ($R1$) and ($R2$) in the Definition 5.4. If each interval point of I is regular and for each endpoint y of I in I, there is an $x \in \mathrm{int}(I)$ such that $\mathbb{P}_x(\tau_y < \infty) > 0$. Then X is a regular diffusion.*

[3] we now reduce the reflected Brownian motion into a "regular" standard Brownian motion. Let the underlying process be B_t instead of X_t, then for all $x \in \mathbb{R}$, x is regular, i.e. $\mathbb{P}_x(\overline{\tau}_x = 0) = 1$, and this gives the final equality in (5.2).

The assumption ($R3$) can be strengthened. For $a < b$, let $J = [a, b]$ be a *closed, infinite, proper interval included in* I. This allows a and b to be endpoints of J in I. Let

$$\tau_J = \tau_{a,b} = \inf\{t > 0 : X_t \in \{a, b\}\} \leq \infty$$

be the hitting time of X of either a or b. Note that $\tau_J = \tau_a \wedge \tau_b$ for $x \in (a, b)$ where x is the starting point of X_t. We start with the following uniform bound.

► **Lemma 5.6** (bounded the hitting time)**.** *There exist K and $\delta > 0$ such that $\mathbb{P}_x(\tau_J \leq K) \geq \delta$ for $x \in J$.*

Proof. Let $y = (a + b)/2$. Since X is regular, there exist sufficiently large $K > 0$ such that,

$$\delta := \max\left(\mathbb{P}_y(\tau_a \leq K), \mathbb{P}_y(\tau_b \leq K)\right) > 0. \tag{5.3}$$

Let the case $a \leq x \leq y$. Then,

$$\tau_a = \tau_x + \tau_a \circ \theta_{\tau_x} \qquad \text{on } X_0 = y.$$

Hence by strong Markov property,

$$\begin{aligned}
\mathbb{P}_y(\tau_a \leq K) &= \mathbb{P}_y(\tau_x + \tau_a \circ \theta_{\tau_x} \leq K) \\
&\leq \mathbb{P}_y(\tau_x < \infty, \tau_a \circ \theta_{\tau_x} \leq K) \\
&= \mathbb{E}_y[\mathbf{1}_{\tau_x < \infty}(\mathbf{1}_{\tau_a \leq K} \circ \theta_{\tau_x})] \\
&= \mathbb{E}_y[\mathbf{1}_{\tau_x < \infty}\, \mathbb{E}_y[\mathbf{1}_{\tau_a \leq K} \circ \theta_{\tau_x} \mid \mathcal{F}_{\tau_x +}]] \\
&= \mathbb{E}_y[\mathbf{1}_{\tau_x < \infty}\, \mathbb{E}_{X_{\tau_x}}[\mathbf{1}_{\tau_a \leq K}]] \\
&= \mathbb{E}_y[\mathbf{1}_{\tau_x < \infty} \mathbb{P}_x(\tau_a \leq K)] \\
&= \mathbb{P}_y(\tau_x < \infty) \mathbb{P}_x(\tau_a \leq K) \\
&\leq \mathbb{P}_x(\tau_a \leq K).
\end{aligned}$$

Hence,

$$\mathbb{P}_x(\tau_J \leq K) \geq \mathbb{P}_x(\tau_a \leq K) \geq \mathbb{P}_y(\tau_a \leq K) \geq \delta.$$ ► not correct. cannot match (5.3).

The other case can be done by symmetry. □

► **Proposition 5.7.** *The hitting time $\tau_J < \infty$ P_x a.s.. Moreover, $\sup_x \mathbb{E}_x[\tau_J] < \infty$.*

Proof. By Lemma 5.6, there exist K and $\delta > 0$ such that

$$\sup_{x \in J} \mathbb{P}_x(\tau_J > K) \leq 1 - \delta < 1.$$

If $\tau_J > s$, we have $\tau_J = s + \tau_J \circ \theta_s$, it follows

$$\begin{aligned}
\mathbb{P}_x(\tau_J > nK) &= \mathbb{P}_x(\tau_J > (n-1)K, \tau_J > nK) \\
&= \mathbb{P}_x(\tau_J > (n-1)K, (n-1)K + \tau_J \circ \theta_{(n-1)K} > nK).
\end{aligned}$$

Since $\{\tau_J > (n-1)K\} \in \mathcal{F}^X_{(n-1)K+}$, by the Markov property,

$$\begin{aligned}
\mathbb{P}_x(\tau_J > nK) &= \mathbb{E}_x[\mathbf{1}_{\tau_J > (n-1)K}(\mathbf{1}_{\tau_J > K} \circ \theta_{(n-1)K})] \\
&= \mathbb{E}_x\left[\mathbf{1}_{\tau_J > (n-1)K}\, \mathbb{E}_x\left[\mathbf{1}_{\tau_J > K} \circ \theta_{(n-1)K} \mid \mathcal{F}^X_{(n-1)K+}\right]\right] \\
&= \mathbb{E}_x\left[\mathbf{1}_{\tau_J > (n-1)K}\, \mathbb{E}_{X_{(n-1)K}}[\mathbf{1}_{\tau_J > K}]\right] \\
&\leq (1-\delta)\, \mathbb{E}_x\left[\mathbf{1}_{\tau_J > (n-1)K}\right] \\
&= (1-\delta) \mathbb{P}_x(\tau_J > (n-1)K).
\end{aligned}$$

Iterating this, we obtain

$$\mathbb{P}_x(\tau_J > nK) \leq (1-\delta)\mathbb{P}_x(\tau_J > (n-1)K) \leq \cdots \leq (1-\delta)^n,$$

for every $x \in J$. Using the well known inequality: let X be a non-negative random variable, then

$$\mathbb{E}[X] \leq \sum_{n=0}^{\infty} \mathbb{P}(X \geq n),$$

we have

$$\mathbb{E}_x\left[\frac{\tau_J}{K}\right] \leq \sum_{n=0}^{\infty} \mathbb{P}_x(\tau_J \geq nK) \leq \sum_{n=0}^{\infty} (1-\delta)^n = \frac{1}{\delta},$$

for all $x \in J$. This shows that

$$\sup_{x \in J} \mathbb{E}_x[\tau_J] \leq \frac{K}{\delta} < \infty,$$

as claimed. In particular, it follows that $\tau_J < \infty$ $\mathbb{P}_x$ a.s.. □

Let X be a one-dimensional regular diffusion. The X can be described in three things

(1) Scale function: describe the diffusion moves to left or right.

(2) Speed measure: the rate at which X leaves some set A.

(3) Boundary behavior.

Moreover, X can be represented as a transformation of BM, where we are transforming space using the scale function and transforming time using speed measure.

5.2 Scale function

▶ **Definition 5.8** (scale function)**.** Let X be a regular diffusion on I. A *scale function* for X is a continuous strictly increasing function $s : I \to \mathbb{R}$ such that $a < x < b \in I$,

$$\mathbb{P}_x(\tau_b < \tau_a) = \frac{s(x) - s(a)}{s(b) - s(a)}. \tag{5.4}$$

If $s(x) = x$ is a scale function for X, we say that X is *in natural scale.*

Aim is to show that for any one-dimensional regular diffusion one can construct a scale function. Fix $a < b$, and let $J = [a, b] \subset I$, define

$$s_J(x) = \mathbb{P}_x(\tau_b < \tau_a), \qquad x \in J. \quad \text{(hitting } b \text{ before hitting } a\text{)}$$

Here, the variable is the starting point of the underlying process. Clearly, $s_J(a) = 0$ and $s_J(b) = 1$. Also, for $a < x < y < b$, by Markov property,

$$s_J(x) = \mathbb{P}_x(\tau_y < \tau_a) s_J(y) \leq s_J(y),$$

so that $s_J(x)$ is non-decreasing for x from a to b.

▶ **Lemma 5.9.** *The function s_J is continuous and strictly increasing.*

▶ **Proposition 5.10** (existence of scale function)**.** *A regular diffusion X has a scale function. This function is unique up to an affine transformation, that is, if $\widetilde{s} : I \to \mathbb{R}$ is another strictly increasing continuous function satisfying*

$$\mathbb{P}_x(\tau_b < \tau_a) = \frac{s(x) - s(a)}{s(b) - s(a)}, \qquad x \in [a, b] \subset I,$$

then there exist $\alpha \neq 0$ and $\beta \in \mathbb{R}$ such that $\widetilde{s}(x) = \alpha s(x) + \beta$ for all $x \in I$.

Proof. If $I = [\ell, r]$ is compact, let $s(x) = s_I(x)$,

$$s_I(x) = \mathbb{P}_x(\tau_r < \tau_\ell).$$

This function is strictly increasing and continuous by Lemma 5.9. To verify that for $x \in [a,b] \subset I = [\ell, r]$,

$$\mathbb{P}_x(\tau_b < \tau_a) = \frac{s_I(x) - s_I(a)}{s_I(b) - s_I(a)},$$

we need the following equality. Let $\ell \le \alpha \le a < x < b \le \beta \le r$,

$$\begin{aligned}\mathbb{P}_x(\tau_\beta < \tau_\alpha) &= \mathbb{E}_x[\mathbf{1}_{\tau_\beta<\tau_\alpha}]\\ \text{(Tower property)} &= \mathbb{E}_x\left[\mathbb{E}_x[\mathbf{1}_{\tau_\beta<\tau_\alpha} \mid \mathcal{F}_{\tau_{a,b}+}]\right]\\ &= \mathbb{E}_x\left[\mathbb{E}_{X_{\tau_{a,b}}}[\mathbf{1}_{\tau_\beta<\tau_\alpha}]\right]\\ &= \mathbb{E}_x\left[\mathbf{1}_{\tau_b<\tau_a}\,\mathbb{E}_b[\mathbf{1}_{\tau_\beta<\tau_\alpha}] + \mathbf{1}_{\tau_a<\tau_b}\,\mathbb{E}_a[\mathbf{1}_{\tau_\beta<\tau_\alpha}]\right],\end{aligned}$$

rewrite this as probability, we have

$$\mathbb{P}_x(\tau_\beta < \tau_\alpha) = \mathbb{P}_x(\tau_b < \tau_a)\mathbb{P}_b(\tau_\beta < \tau_\alpha) + \mathbb{P}_x(\tau_a < \tau_b)\mathbb{P}_a(\tau_\beta < \tau_\alpha). \tag{5.5}$$

By taking $\alpha = \ell$, and $\beta = r$, we have

$$\mathbb{P}_x(\tau_r < \tau_\ell) = \mathbb{P}_x(\tau_b < \tau_a)\mathbb{P}_b(\tau_r < \tau_\ell) + \mathbb{P}_x(\tau_a < \tau_b)\mathbb{P}_a(\tau_r < \tau_\ell).$$

which gives in terms of function s_I, we have

$$s_I(x) = \mathbb{P}_x(\tau_b < \tau_a)s_I(b) + \mathbb{P}_x(\tau_a < \tau_b)s_I(a).$$

Taking into the account that

$$1 = \mathbb{P}_x(\tau_{\alpha,\beta} < \infty) = \mathbb{P}_x(\tau_b < \tau_a) + \mathbb{P}_x(\tau_a < \tau_b),$$

then by solving these linear equations, we have

$$\mathbb{P}_x(\tau_b < \tau_a) = \frac{s_I(x) - s_I(a)}{s_I(b) - s_I(a)}.$$

We have now proved the existence of a scale function when the state space I is compact. The problem is $s_I(x)$ depends on I, in particular, it doesn't make sense that it is not compact. E.g. consider the SBM in Example 5.3, $I = (-\infty, \infty)$, then $s_I(x) = \mathbb{P}_x(\tau_\infty < \tau_{-\infty})$ is not making any sense.

If I is not compact, then let consider an *increasing sequence of interval* $J_n = [a_n, b_n]$ such that $J_n \uparrow I$. In particular, if I contains left endpoint ℓ (right endpoint r), then we put $a_n = \ell$ ($b_n = r$) for all $n \ge 1$.

Let us define a sequence of functions $s^{(n)} : J_n \to \mathbb{R}$ inductively as follows: let $s^{(1)} = s_{J_1}$ and for $n \ge 1$,

$$s^{(n+1)}(x) = \left(s^{(n)}(b_n) - s^{(n)}(a_n)\right)\frac{s_{J_{n+1}}(x) - s_{J_{n+1}}(a_n)}{s_{J_{n+1}}(b_n) - s_{J_{n+1}}(a_n)} + s^{(n)}(a_n). \tag{5.6}$$

Notice that, $s^{(n+1)}(x)$ is an affine transformation of $s_{J_{n+1}}(x)$. Moreover, it follows the definition that

$$s^{(n+1)}(a_n) = s^{(n)}(a_n), \quad s^{(n+1)}(b_n) = s^{(n)}(b_n), \qquad n \ge 1. \tag{5.7}$$

Now, we will show that these functions, $s^{(n)}$, are the same inside J_n. Namely,

$$s^{(n+1)}(x) = s^{(n)}(x), \qquad \forall x \in J_n. \tag{5.8}$$

Let $\alpha = a_{n+1}$, $\beta = b_{n+1}$, and $a = a_n$, $b = b_n$ in (5.5), we have

$$\begin{aligned}s_{J_{n+1}}(x) &= \mathbb{P}_x(\tau_{b_{n+1}} < \tau_{a_{n+1}})\\ &= \mathbb{P}_x(\tau_{a_n} < \tau_{b_n})\mathbb{P}_{a_n}(\tau_{b_{n+1}} < \tau_{a_{n+1}}) + \mathbb{P}_x(\tau_{b_n<\tau_{a_n}})\mathbb{P}_{b_n}(\tau_{b_{n+1}} < \tau_{a_{n+1}})\\ &= (1 - s_{J_n}(x))s_{J_{n+1}}(a_n) + s_{J_n}(x)s_{J_{n+1}}(b_n).\end{aligned}$$

Plug this into (5.6), we have

$$s^{(n+1)}(x) = s^{(n)}(a_n) + (s^{(n)}(b_n) - s^{(n)}(a_n))s_{J_n}(x) = c_1 s_{J_n}(x) + c_2, \qquad x \in J_n.$$

Moreover,

$$s^{(n)}(x) = \widetilde{c}_1 s_{J_n}(x) + \widetilde{c}_2.$$

From (5.7), we acquire $\widetilde{c}_1 = c_1$ and $\widetilde{c}_2 = c_2$, which proves (5.8). *Up to now, we have proved the function s_{J_n} exist and if we pass the limit $n \to \infty$, we can get a scale function for all $x \in I$ since $J_n \uparrow I$, and one scale function is enough since they are coincide with the most general one.* □

▶ **Example 5.11.** Consider a standard Brownian motion $B = (B_t)_{t\geq 0}$. It was shown that

$$\mathbb{P}_x(\tau_b < \tau_a) = \frac{x-a}{b-a}.$$

Hence, the scale function of a SBM is $s(x) = x$. Thus, SBM is in natural scale.

▶ **Remark.** The scale function describe the tendency of the process to move to the "right" and to the "left". [4] This is seen from (5.4) as follows. From (5.4), we have

$$\mathbb{P}_x(\tau_a < \tau_b) = \frac{s(b) - s(x)}{s(b) - s(a)}$$

Thus, setting $x = (a+b)/2$, it is seen that

$$\mathbb{P}_x(\tau_a < \tau_b) \geq \mathbb{P}_x(\tau_b < \tau_a)$$

if and only if $s(x) \leq (s(a) + s(b))/2$. Thus if s is convex at a point, then X tends to move more to the lower boundary. Likewise, if s is concave at a point, then X tends to move more to the upper boundary.

▶ **Proposition 5.12.** *Let $X = (X_t)_{t\geq 0}$ be a regular diffusion. Then the process $s(X) = (s(X_t))_{t\geq 0}$ is a regular diffusion and is in natural scale.*

▶ **Lemma 5.13.** *Let $J = [a,b] \subset I$. Then for $x \in J$, $s_J(X_{t\wedge\tau_{a,b}})$ is a continuous $\mathbb{P}_x$-martingale.*

Proof. To prove the martingale property, we first prove the following,

$$s_J(X_{t\wedge\tau_{a,b}}) = \mathbb{P}_x(\tau_b < \tau_a \mid \mathcal{F}^X_{t+}).$$

Note,

$$\begin{aligned}\mathbb{P}_x(\tau_b < \tau_a \mid \mathcal{F}^X_{t+}) &= \mathbb{E}_x[\mathbf{1}_{\tau_b<\tau_a}\mathbf{1}_{\tau_b\leq t} \mid \mathcal{F}^X_{t+}] + \mathbb{E}_x[\mathbf{1}_{\tau_b<\tau_a}\mathbf{1}_{\tau_b>t} \mid \mathcal{F}^X_{t+}] \\ &= \mathbb{E}_x[\mathbf{1}_{\tau_b<\tau_a}\mathbf{1}_{\tau_b\leq t} \mid \mathcal{F}^X_{t+}] + \mathbb{E}_x[\mathbf{1}_{\tau_b<\tau_a}\mathbf{1}_{\tau_b>t}\mathbf{1}_{\tau_a>t} \mid \mathcal{F}^X_{t+}]\end{aligned}$$

Moreover, the event

$$\{\tau_b < \tau_a\} \cup \{\tau_b \leq t\} = (\cup_{s\in\mathbb{Q},\ s\leq t}\{\tau_b < s < \tau_a\} \cap \{\tau_a \leq t\}) \cup \{\tau_b \leq t < \tau_a\}$$

is $\mathcal{F}^X_{t+}$ measurable. Therefore

$$\begin{aligned}\mathbb{P}_x(\tau_b < \tau_a \mid \mathcal{F}^X_{t+}) &= \mathbf{1}_{t\geq\tau_b,\tau_a>\tau_b} + \mathbf{1}_{\tau_b>t,\tau_a>t}\,\mathbb{E}_{X_t}[\mathbf{1}_{\tau_b<\tau_a}] \\ &= \mathbf{1}_{t\geq\tau_{a,b}}\mathbf{1}_{\tau_a>\tau_b} + \mathbf{1}_{\tau_b>t,\tau_a>t}s_J(X_t) = s_J(X_{s\wedge\tau_{a,b}}).\end{aligned}$$

Then, for $s \leq t$,

$$\mathbb{E}_x[s_J(X_{t\wedge\tau_{a,b}}) \mid \mathcal{F}^X_{s+}] = \mathbb{E}_x[\mathbb{P}_x(\tau_b \leq \tau_a \mid \mathcal{F}^X_{t+}) \mid \mathcal{F}^X_{s+}] = \mathbb{P}_x(\tau_b < \tau_a \mid \mathcal{F}^X_{s+}) = s_J(X_{s\wedge\tau_{a,b}}).$$

□

▶ **Definition 5.14** (local martingale)**.** An adapted (right) continuous process $M = (M_t)_{t\geq 0}$ (with $M_0 \in L^1$ or else $M := M - M_0$) is called a *local martingale*, if there exists a sequence of stopping time $\{\tau_n\}_{n\geq 1}$ such that

(1) $\tau_n \uparrow \infty$, as $n \to \infty$.

(2) $X^{\tau_n} = (X^{\tau_n}_t)_{t\geq 0} = (X_{t\wedge\tau_n})_{t\geq 0}$ is a martingale for each $n \geq 1$.

▶ **Remark.** The sequence $\{\tau_n\}_{n\geq 1}$ is called a *localization sequence*. Each martingale is a local martingale and the converse is not true in general. A typical localization sequence is given by

$$\tau_n = \inf\{t > 0 : |\, M_n \,| = n\}, \qquad n \geq 1.$$

Note that M^{τ_n} is bounded, i.e. $|\, M_{t\wedge\tau_n} \,| \leq n$ for all $t \geq 0$ when $n \geq 1$ is fixed.

[4] The quotation mark here is to indicate that the left and right mentioned here are in the state space I, not in time space $\mathbb{R}_+$.

▶ **Proposition 5.15.** *A locally bounded measurable function $f : I \to \mathbb{R}$ is a scale function of X if and only if $f(X)^{\tau_{\ell,r}} = (f(X_{t\wedge\tau_{\ell,r}}))_{t\geq 0}$ is a local martingale.*

Proof. ($\Leftarrow$): if $f(X)^{\tau_{\ell,r}}$ is a local martingale. Let $x \in (a,b)$, and using

$$|\, f(X_t)^{\tau_{a,b}}\,| = |\, f(X_{t\wedge\tau_{a,b}})\,| \leq C$$

since f is locally bounded. Therefore $f(X_t)^{\tau_{a,b}}$ is a bounded martingale. Thus

$$\begin{aligned} f(x) &= \mathbb{E}_x[f(X_0)] = \mathbb{E}_x[f(X_{0\wedge\tau_{a,b}})] = \mathbb{E}_x[f(X_{\tau_{a,b}})] \\ &= \mathbb{E}_x[\mathbf{1}_{\tau_a<\tau_b} f(X_{\tau_{a,b}})] + \mathbb{E}_x[\mathbf{1}_{\tau_b<\tau_a} f(X_{\tau_{a,b}})] \\ &= f(a)\mathbb{P}_x(\tau_a<\tau_b) + f(b)\mathbb{P}_x(\tau_b<\tau_a). \end{aligned}$$

On the other hand, we know that

$$1 = \mathbb{P}_x(\tau_a<\tau_b) + \mathbb{P}_x(\tau_b<\tau_a).$$

Solving the linear system, we have

$$\mathbb{P}_x(\tau_b<\tau_a) = \frac{f(x)-f(a)}{f(b)-f(a)}$$

showing that f is a scale function. ◻

5.3 Green Functions

▶ **Definition 5.16** (convex function)**.** A function is *convex* on a convex set C, if for any $x, y \in C$, and λ, μ such that $\lambda + \mu = 1$, the following inequality holds

$$f(\lambda x + \mu y) \leq \lambda f(x) + \mu f(y).$$

We say that function f is *concave* on a set C if function $-f$ is convex.

Fixed $a < b$. Let function f be convex on $[a,b]$. Observe that for any $a < x \leq y \leq z < b$, the following inequality holds

$$f(y) = f\left(\frac{z-y}{z-x}x + \frac{y-x}{z-x}z\right) \leq \frac{z-y}{z-x}f(x) + \frac{y-x}{z-x}f(z). \tag{5.9}$$

Subtracting $f(x)$ from both sides and dividing by $y - x$, we obtain

$$\frac{f(y)-f(x)}{y-x} \leq \frac{f(z)-f(x)}{z-x}.$$

Therefore the function

$$z \mapsto \frac{f(z)-f(x)}{z-x}$$

is an increasing function in z. Rearranging (5.9), we have

$$\frac{f(y)-f(x)}{y-x} \leq \frac{f(z)-f(y)}{z-y},$$

which implies that

$$z \mapsto \frac{f(z)-f(x)}{z-x}$$

is bounded from below. Hence the right derivative at x exist

$$f^+(x) := \lim_{y\downarrow x} \frac{f(y)-f(x)}{y-x}.$$

Similarly, the left derivative at x exists as well

$$f^-(x) = \lim_{z\uparrow x} \frac{f(z)-f(x)}{z-x} \leq f^+(x).$$

Moreover, we have the following properties of these left and right derivatives,

$(M1)$ $x \mapsto f^+(x)$ and $x \mapsto f^-(x)$ are increasing

$(M2)$ $x \mapsto f^+(x)$ is right continuous.

$(M3)$ $x \mapsto f^-(x)$ is left continuous.

$(M4)$ The set $\{x \in (a,b) : f^+(x) \neq f^-(x)\}$ is at most countable.

From $(M1)$ and $(M2)$, there exist a unique non-negative locally finite measure μ on the interval (a,b) associated with $x \mapsto f^+(x)$ by means of

$$\mu((c,d]) = f^+(d) - f^+(c), \qquad (c,d] \subset (a,b). \tag{5.10}$$

The measure μ is defined on $\mathcal{B}((a,b))$.

▶ **Remark.** $f^+(x)$ is increasing on (a,b) is equivalent to convexity. The sufficient condition for convexity will be the second derivative $f''(x) \geq 0$. E.g. $f(x) = x^2$ is a convex functions, and $f''(x) = 2 > 0$. This allow us to simplify the expression for measure μ under additional assumption that f is twice continuously differentiable. In this case, $f^+ = f'$, and (5.10) becomes

$$\mu((c,d]) = f'(d) - f'(c) = \int_c^d f''(z)\, \mathrm{d}z,$$

where $f'' > 0$ on (a,b). Hence μ is absolutely continuous with the density given by f''. Namely, for any compact subset B of (a,b),

$$\mu(B) = \int_B f''(z)\, \mathrm{d}z.$$

▶ **Definition 5.17** (Green function)**.** The *Green function* $G_{a,b} : [a,b] \times [a,b] \mapsto \mathbb{R}$ is defined as follows

$$G_{a,b}(x,y) = \begin{cases} \frac{(x-a)(b-y)}{b-a} & \text{for } a \leq x \leq y \leq b. \\ \frac{(y-a)(b-x)}{b-a} & \text{for } a \leq y \leq x \leq b. \end{cases}$$

▶ **Lemma 5.18** (convexity and Green function)**.** *Let f be convex on (a,b) and continuous on $[a,b]$ with $f(a) = f(b) = 0$. Then*

$$f(x) = -\int_{(a,b)} G_{a,b}(x,y)\mu(\mathrm{d}y), \qquad x \in (a,b),$$

where μ is defined in (5.10)*.*

Proof. First for $a < x \leq y < b$. Then

$$f(y) - f(x) = \int_{(x,y)} f^+(z)\, \mathrm{d}z.$$

Let $y \uparrow b$ and using the continuity of f and monotonicity of $f^+(z)$, we obtain

$$f(b) - f(x) = \int_{(x,b)} f^+(z)\, \mathrm{d}z.$$

Using (5.10) and Fubini theorem with the assumption that $f(b) = 0$, we have

$$\begin{aligned} -f(x) &= \int_{(x,b)} \left(f^+(x) + \int_{(x,z]} \mu(\mathrm{d}w) \right) \mathrm{d}z \\ &= (b-x)f^+(x) + \int_{(x,b)} \int_{(w,b)} \mathrm{d}z\ \mu(\mathrm{d}w) \\ &= (b-x)f^+(x) + \int_{(x,b)} (b-w)\ \mu(\mathrm{d}w). \end{aligned}$$

Similarly,

$$f(x) = (x-a)f^+(x) - \int_{(a,x)} (w-a)\ \mu(\mathrm{d}w).$$

Multiply the first equality by $(a-x)$, the second by $(b-x)$ and adding them to get

$$(b-a)f(x) = -(x-a)\int_{(x,b)}(b-w)\ \mu(\mathrm{d}w) - (b-x)\int_{(a,x)}(w-a)\ \mu(\mathrm{d}w). \qquad \square$$

□

The analysis can be extend to concave functions. For example, we can define the associated measure, taking into account that $x \mapsto f^+(x)$ is decreasing,

$$\mu_{\text{concave}}((c,d]) = f^+(c) - f^+(d), \qquad (c,d] \subset (a,b). \tag{5.11}$$

This measure is defined on $\mathcal{B}((a,b))$. Therefore we have a lemma,

▶ **Lemma 5.19.** *Let f be concave on (a,b), continuous on $[a,b]$ with $f(a) = f(b) = 0$. Then*

$$f(x) = \int_{(a,b)} G(x,y)\ \mu(\mathrm{d}y), \qquad x \in [a,b],$$

where μ is defined by (5.11).

5.4 Speed Measure

We turn to another important characteristic of the diffusion process called the *speed measure*. We will assume that X is a regular diffusion in natural scale. If X is not natural scale, we can transform it into natural scale by Proposition 5.12.

Let $J = [a,b] \subset I$. Recall that $\tau_J = \tau_{a,b} \le \infty$ is the first hitting time of either a or b. Put

$$m_J(x) := m_{a,b}(x) := \mathbb{E}_x[\tau_J].$$

Then $m_J(x) < J$ for $x \in J$ by Proposition 5.7. Since X is a regular diffusion $m_J(a) = m_J(b) = 0$.

▶ **Theorem 5.20.** *Function m_J is continuous and strictly concave on $[a,b]$.*

Proof. Consider the points $a \le c \le x \le d \le b$. By the strong Markov property of X, we have

$$m_{a,b}(x) = m_{c,d}(x) + \frac{d-x}{d-c}m_{a,b}(c) + \frac{x-c}{d-c}m_{a,b}(d).$$

Observe that, we can put

$$x = \lambda c + \mu d, \qquad \lambda = \frac{d-x}{d-c}, \qquad \mu = \frac{x-c}{d-c}.$$

It is clear that $\lambda, \mu \ge 0$, and $\lambda + \mu = 1$. Then, using the fact that $m_{c,d}(x) > 0$ for $x \in (c,d)$, we have

$$m_{a,b}(\lambda c + \mu d) > \frac{d-x}{d-c}m_{a,b}(c) + \frac{x-c}{d-c}m_{a,b}(d) = \lambda m_{a,b}(c) + \mu m_{a,b}(d).$$

Therefore $m_{a,b}(x)$ is concave on $[a,b]$. □

▶ **Theorem 5.21.** *There exist a unique non-negative locally finite measure m defined on $\mathcal{B}((\ell,r))$ such that for any $[a,b] \subset [\ell,r]$, we have*

$$m_{a,b}(x) = \int_{(a,b)} G_{a,b}(x,y)\ m(\mathrm{d}y), \qquad x \in [a,b]. \tag{5.12}$$

▶ **Definition 5.22** (speed measure)**.** The measure m defined in Theorem 5.21 is called the *speed measure* of the process X.

▶ **Example 5.23.** Consider a SBM $B = (B_t)_{t\ge 0}$. Brownian motion is already in natural scale. To find the speed measure of B, it is more convenient to use (5.11).

$$m_{a,b}(x) = \mathbb{E}_x[\tau_{a,b}] = \frac{a^2(b-x) + b^2(x-a)}{b-a} - x^2.$$

This function is continuously differentiable and hence the right and left derivatives are the same,

$$m_{a,b}^{+}(x) = \frac{\mathrm{d}m_{a,b}(x)}{\mathrm{d}x} = -2x - \frac{b^2 - a^2}{b-a} = -2x - (b+a).$$

Then for any $(c,d] \subset (a,b]$, we have

$$m_{a,b}((c,d]) = 2(d-c).$$

Hence $m_{a,b}((c,d])$ is absolutely continuous and

$$m(\mathrm{d}x) = 2\lambda(\mathrm{d}x),$$

where λ is the Lebesgue measure on $\mathcal{B}(\mathbb{R})$.

▶ **Corollary 5.24.** *If $f : \mathbb{R} \to \mathbb{R}$ is a (non-negative or bounded) Borel function, then we have*

$$\mathbb{E}_x\left[\int_0^{\tau_{a,b}} f(X_t)\ \mathrm{d}t\right] = \int_a^b f(y)G_{a,b}(x,y)\ m(\mathrm{d}y), \qquad x \in (a,b) \subset I. \tag{5.13}$$

Proof. Note that (5.13) reduce to (5.12) for $f \equiv 1$. To prove for general Borel function, it is sufficient to prove for $f = \mathbf{1}_{(c,b)}$ for $c \in (a,b)$. Define the following function

$$v(x) = \mathbb{E}_x\left(\int_0^{\tau_{a,b}} \mathbf{1}_{(c,b)}(X_t)\ \mathrm{d}t\right), \qquad x \in (a,b).$$

By argument from previous theorem, we have $v(x)$ is concave. Moreover, by strong Markov property,

$$v(x) = \begin{cases} \mathbb{E}_x[\tau_{c,b}] + \left(\frac{b-x}{b-c}\right) v(c) & \text{if } x \in (c,b), \\ \frac{x-a}{c-a} v(c) & \text{if } x \in (a,c]. \end{cases}$$

Now, in particular, v is continuous, and $v(b) = v(a) = 0$. Therefore, by Lemma 5.19

$$v(x) = \int_a^b G_{a,b}(x,y)\ m_v(\mathrm{d}y), \qquad x \in (a,b).$$

Using the definition of m_v, we have

$$m_v((\alpha,\beta]) = v^+(\alpha) - v^+(\beta) = m((\alpha,\beta) \cap (c,b)).$$

So, $m_v(\mathrm{d}y) = \mathbf{1}_{(c,d)}(y)\ m(\mathrm{d}y)$. So for $f = \mathbf{1}_{(c,b)}$,

$$v(x) = \mathbb{E}_x\left[\int_0^{\tau_{a,b}} f(X_t)\ \mathrm{d}t\right] = \int_c^d G_{a,b}(x,y)\mathbf{1}_{(c,d)}(y)\ m(\mathrm{d}y).$$

Extension to general Borel function can be done by using linearity. □

▶ **Remark.** If X is not in natural scale, it is also possible to define a speed measure. Let s be a scale function of X. Introduce a general form of the Green function

$$G_{a,b}(x,y) = \begin{cases} \dfrac{(s(b)-s(y))(s(x)-s(a))}{s(b)-s(a)}, & \text{for } a \le x \le y \le b, \\ \dfrac{(s(y)-s(a))(s(b)-s(x))}{s(b)-s(a)} & \text{for } a \le y \le x \le b. \end{cases} \tag{5.14}$$

Therefore by Theorem 5.21 and Corollary 5.24, we have

▶ **Theorem 5.25.** *Let X be a one-dimensional regular diffusion with scale function s, then*

(1) *There exist a unique* Lebesgue-Stiltjes measure *m on $\mathcal{B}((\ell,r))$ such that for any $[a,b] \subset (\ell,r)$, we have*

$$m_{a,b}(x) = \int_a^b G_{a,b}(x,y)\ m(\mathrm{d}y), \qquad x \in [a,b],$$

where $G_{a,b}$ is defined in (5.14).

(2) *Moreover, if $f : \mathbb{R} \to \mathbb{R}$ is a (non-negative or bounded) Borel function, then we have*

$$\mathbb{E}_x\left[\int_0^{\tau_{a,b}} f(X_t)\ \mathrm{d}t\right] = \int_a^b f(y)G_{a,b}(x,y)\ m(\mathrm{d}y), \qquad x \in [a,b] \subset (\ell,r),$$

where $G_{a,b}$ is given in (5.14).

5.5 Infinitesimal Generator

Recall that $C_b(I)$ is a norm space that consists of function which are bounded and continuous in I with the supermum norm $\| f \| = \sup_{x\in I} | f(x) |$. Also, for a strong Markov process, we can define a family of transition operators $(P_t)_{t\geq 0}$

$$P_t f(x) = \mathbb{E}_x[f(X_t)] = \int_I P_t(x, \mathrm{d}y) f(y).$$

▶ **Proposition 5.26** (continuity property)**.** *Let X be a regular one-dimensional diffusion with transition operators $(P_t)_{t\geq 0}$ and $f \in C_b(I)$. Then function the mapping $(t, x) \mapsto P_t f(x)$ is jointly continuous on $[0, \infty) \times I$. In particular, X is a Feller process.*

Proof. Fix $f \in C_b(I)$ and $t \in [0, \infty)$. Consider two sequences $t_n \to t$ and $x_n \to x \in \mathrm{int}(I) = (\ell, r)$. Since X is continuous and all $x \in (\ell, r)$ is regular, we obtain

$$\mathbb{P}_x(\tau_{x_n} \to 0) = 1, \quad \mathbf{1}_{\tau_{x_n}<\infty} \to 1, \qquad n \to \infty.$$

By continuity, $X_{\tau_{x_n}+t_n} \to X_t$ as $n \to \infty$. Since f is bounded and continuous, we have by dominated convergence theorem,

$$\mathbb{E}_x[f(X_{\tau_{x_n}+t_n}\mathbf{1}_{\tau_{x_n}<\infty}] \to \mathbb{E}_x\, f(X_t) = P_t f(x) = P_t f(x), \tag{5.15}$$

the last equality is the definition of the transition operator (4.4). By the strong Markov property, we have

$$\begin{aligned}
\mathbb{E}_x[f(X_{\tau_{x_n}+t_n})\mathbf{1}_{\tau_{x_n}<\infty}] &= \mathbb{E}_x\left[\mathbb{E}_x[f(X_{\tau_{x_n}+t_n})\mathbf{1}_{\tau_{x_n}<\infty} \mid \mathcal{F}_{\tau_{x_n}+}]\right] \\
&= \mathbb{E}_x\left[\mathbf{1}_{\tau_{x_n}<\infty}\, \mathbb{E}_x[f(X_{t_n}) \circ \theta_{\tau_{x_n}} \mid \mathcal{F}_{\tau_{x_n}+}]\right] \\
\text{(Strong Markov Property)} \quad &= \mathbb{E}_x\left[\mathbf{1}_{\tau_{x_n}<\infty}\, \mathbb{E}_{X_{\tau_{x_n}}}[f(X_{t_n})]\right] \\
&= P_{t_n} f(x_n)\mathbb{P}_x(\tau_{x_n} < \infty)
\end{aligned}$$

Taking the limit and using the dominant convergence theorem, we have

$$\lim_{n\to\infty} P_{t_n} f(x_n)\mathbb{P}_x(\tau_{x_n} < \infty) \to P_t f(x).$$

Hence we proved (5.15). ∎

Thus we can see that for regular one-dimensional diffusion, the class of Feller processes coincided with the class of strong Markov process.

▶ **Definition 5.27** (resolvent)**.** For $\lambda > 0$ define the *resolvent operator* $R_\lambda : C_b(I) \to C_b(I)$ as,

$$R_\lambda f(x) = \int_0^\infty \mathrm{e}^{-\lambda t} P_t f(x)\ \mathrm{d}t.$$

▶ **Proposition 5.28** (properties of resolvent)**.** *Let $\lambda, \mu > 0$, then*

(1) *The resolvent equation holds, that is*

$$R_\mu = R_\lambda + (\lambda - \mu)R_\lambda R_\mu.$$

(2) $R_\lambda R_\mu = R_\mu R_\lambda$.

(3) $R_\lambda f \equiv 0$ *if and only if* $R_\mu f \equiv 0$.

(4) $R_\lambda f \equiv 0$ *if and only if* $f \equiv 0$.

(5) *The range of $R_\lambda f$ does not depend on λ.*

Proof of (4). If $f \equiv 0$ then $R_\lambda f \equiv 0$ by definition. Therefore, let $R_\lambda f \equiv 0$. Then for any $\mu > 0$, by the resolvent equation

$$\begin{aligned} R_\mu f &= R_\lambda f + (\lambda - \mu) R_\lambda R_\mu f \quad \text{(resolvent equation)} \\ &= R_\lambda f + (\lambda - \mu) R_\mu R_\lambda f \equiv 0 \quad \text{(commutativity)}. \end{aligned}$$

Note that

$$\begin{aligned} 0 = R_\mu f &= \int_0^\infty \mathrm{e}^{-\mu t} P_t f(x)\, \mathrm{d}t \\ &= \frac{1}{\mu} \int_0^\infty \mathrm{e}^{-s} P_{s/\mu} f(x)\, \mathrm{d}s \quad \text{(change of variable, } s = \mu t) \end{aligned}$$

Since $\lambda > 0$, we have

$$\int_0^\infty \mathrm{e}^{-s} P_{s/\mu} f(x)\, \mathrm{d}s = 0.$$

Notice, by Proposition 5.26, we have

$$P_{s/\mu} f(x) \to P_0 f(x) \qquad \mu \to \infty.$$

Moreover, since X is a Feller process, therefore $P_{t/\mu} f(x) \le \| f \|$. Therefore, we have, by dominated convergence theorem

$$\begin{aligned} &\lim_{\mu\to\infty} \int_0^\infty \mathrm{e}^{-s} P_{s/\mu} f(x)\, \mathrm{d}s \\ &= \int_0^\infty \lim_{\mu\to\infty} \mathrm{e}^{-s} P_{s/\mu} f(x)\, \mathrm{d}s \quad \text{(dominated convergence theorem)} \\ &= \int_0^\infty \mathrm{e}^{-s} P_0 f(x)\, \mathrm{d}s = f(x) \equiv 0. \quad \left(\text{use } \int_0^\infty \mathrm{e}^{-s}\, \mathrm{d}s = 1\right) \end{aligned}$$

□

Proof of (5). Let $f = R_\mu g$. The statement is equivalent to: there exist $\widetilde{g}$ such that $f = R_\lambda \widetilde{g}$. It follows from the resolvent equation that

$$f = R_\mu g = R_\lambda g + (\lambda - \mu) R_\lambda R_\mu g = R_\lambda (g + (\lambda - \mu) R_\mu g).$$

Hence, we can simply set $\widetilde{g} = g + (\lambda - \mu) R_\mu g$. □

Let id be the identity operator on $C_b(I)$ given by $\mathrm{id} f = f$. It follows from (4) of Proposition 5.28 that $R_\lambda f \equiv 0$ if and only if $f \equiv 0$. This implies that the inverse $R_\lambda^{-1} f$ is well defined for any f from the range of R_λ.

▶ **Definition 5.29** (infinitesimal generator)**.** Let us denote the domain of the operator A by

$$\mathbb{D}_A = \{ f \mid f = R_\lambda g \text{ for some } g \in C_b(I) \}.$$

For any $f \in \mathbb{D}_A$, define

$$Af = f - R_1^{-1} f.$$

We call $(A, \mathbb{D}_A)$ the infinitesimal generator of $(P_t)_{t \ge 0}$ with the domain $\mathbb{D}_A$. Or we say *A is an infinitesimal generator of X with domain $\mathbb{D}_A$.*

This definition is unclear at the moment, we will show in the next several result that the infinitesimal generator A fully characterize the behavior of the diffusion X at short time scale. Using the resolvent equation, one can show that, using $R_\mu^{-1} = R_\lambda^{-1} - \lambda + \mu$,

$$R_\mu^{-1} = \mu - A \tag{5.16}$$

▶ **Theorem 5.30.** *Infinitesimal generator $(A, \mathbb{D}_A)$ determines the law of X.*

Proof. Given A, we can first define R_μ^{-1} for all μ by (5.16). Then we can define R_μ since all the resolvent are well defined in sense of they all have inverse. Since $R_\mu f$ is a Laplace transform of $P_t f$, we can invert the Laplace transform and obtain $P_t f$. Thus, we can obtain the transition probabilities and hence we can define the law of X by Proposition 4.4. □

▶ **Lemma 5.31** (Dynkin's formula). *Let X be a regular one-dimensional diffusion with infinitesimal generator $(A, \mathbb{D}_A)$. Let τ be a stopping time with finite mean $\mathbb{E}_x[\tau] < \infty$. Then for any $f \in \mathbb{D}_A$, we have*

$$\mathbb{E}_x[f(X_\tau)] = f(x) + \mathbb{E}_x\left[\int_0^\tau Af(X_t)\ \mathrm{d}t\right].$$

Proof. Since $f \in \mathbb{D}_A$, there exist $g \in C_b(I)$ such that $f = R_\lambda g$. Then,

$$\begin{aligned} f(x) = R_\lambda g(x) &= \int_0^\infty \mathrm{e}^{-\lambda t} P_t g(x)\ \mathrm{d}t \\ &= \int_0^\infty \mathrm{e}^{-\lambda t}\, \mathbb{E}_x[g(X_t)]\ \mathrm{d}t \qquad \text{(definition of } P_t) \\ &= \mathbb{E}_x\left[\int_0^\infty \mathrm{e}^{-\lambda t} g(X_t)\ \mathrm{d}t\right] \qquad \text{(Fubini Theorem)} \\ &= \mathbb{E}_x\left[\int_0^\tau \mathrm{e}^{-\lambda t} g(X_t)\ \mathrm{d}t\right] + \mathbb{E}_x\left[\int_\tau^\infty \mathrm{e}^{-\lambda t} g(X_t)\ \mathrm{d}t\right] \end{aligned}$$

Second term can be simplified using the strong Markov property,

$$\begin{aligned} &\mathbb{E}_x\left[\int_\tau^\infty \mathrm{e}^{-\lambda t} g(X_t)\ \mathrm{d}t\right] \\ =&\mathbb{E}_x\left[\mathrm{e}^{-\lambda\tau}\int_0^\infty \mathrm{e}^{-\lambda s} g(X_{s+\tau}\ \mathrm{d}s\right] \qquad \text{(change of variable, } t = s+\tau) \\ =&\mathbb{E}_x\left[\mathrm{e}^{-\lambda\tau}\, \mathbb{E}_x\left[\int_0^\infty \mathrm{e}^{-\lambda s} g(X_{s+\tau})\ \mathrm{d}s \mid \mathcal{F}_{\tau+}\right]\right] \qquad \text{(Tower property)} \\ =&\mathbb{E}_x\left[\mathrm{e}^{-\lambda\tau}\, \mathbb{E}_{X_\tau}\left[\int_0^\infty \mathrm{e}^{-\lambda s} g(X_s)\ \mathrm{d}s\right]\right] \qquad \text{(Strong Markov property)} \\ =&\mathbb{E}_x\left[\mathrm{e}^{-\lambda\tau}\int_0^\infty \mathrm{e}^{-\lambda s} P_s g(X_\tau)\ \mathrm{d}s\right] \qquad \text{(Fubini and definition of } P_t) \\ =&\mathbb{E}_x\left[\mathrm{e}^{-\lambda\tau} R_\lambda g(X_\tau)\right] \qquad \text{(definition of } R_\lambda). \end{aligned}$$

Hence for $f \in \mathbb{D}_A$, we have

$$f(x) = \mathbb{E}_x\left[\int_0^\tau \mathrm{e}^{-\lambda t} g(X_t)\ \mathrm{d}t\right] + \mathbb{E}_x\left[\mathrm{e}^{-\lambda\tau} R_\lambda g(X_\tau)\right].$$

For any λ, there exist a function $g_\lambda \in C_b(I)$ such that $f = R_\lambda g_\lambda$, or equivalently by (5.16), $g_\lambda = (\lambda - A)f$. Then

$$\begin{aligned} f(x) &= \mathbb{E}_x\left[\int_0^\tau \mathrm{e}^{-\lambda t}(\lambda f(X_t) - Af(X_t))\ \mathrm{d}t\right] + \mathbb{E}_x[\mathrm{e}^{-\lambda\tau} f(X_\tau)] \\ &= \mathbb{E}_x\left[\int_0^\tau \lambda \mathrm{e}^{-\lambda t} f(X_t)\ \mathrm{d}t\right] - \mathbb{E}_x\left[\int_0^\tau \mathrm{e}^{-\lambda t} Af(X_t)\ \mathrm{d}t\right] + \mathbb{E}_x\left[\mathrm{e}^{-\lambda\tau} f(X_\tau)\right]. \end{aligned}$$

Letting $\lambda \to 0$, we obtain by dominated convergence theorem,

$$\mathbb{E}_x\left[\mathrm{e}^{-\lambda\tau} f(X_\tau)\right] \to \mathbb{E}_x\left[f(X_\tau)\right].$$

Moreover, since $Af \in C_b(I)$ is bounded and $\mathbb{E}[\tau] < \infty$, apply the dominated convergence theorem again we have

$$\lim_{\lambda\to 0} \mathbb{E}_x\left[\int_0^\tau \mathrm{e}^{-\lambda t} Af(X_t)\ \mathrm{d}t\right] = \mathbb{E}_x\left[\int_0^\tau Af(X_t)\ \mathrm{d}t\right]$$

and

$$\lim_{\lambda\to 0}\mathbb{E}_x\left[\int_0^\tau \lambda e^{-\lambda t} f(X_t)\,dt\right] = \lim_{\lambda\to 0}\mathbb{E}_x\left[\lambda\int_0^\infty e^{-\lambda t}\underbrace{f(X_t)\mathbf{1}_{t\le\tau}}_{\le\| f \|\,\mathbb{E}[\tau]}\,dt\right] = 0,$$

This proves the Lemma. □

Now, we present an equivalent definition of the infinitesimal generator. First, we need the following,

▶ **Definition 5.32.** We say that functions ϕ_t converges boundedly pointwise to some $\phi(x)$ on some subset C of their domains as $t\to 0$ if

(1) Pointwise limit exist.

$$\lim_{t\to 0}\phi_t(x) = \phi(x), \quad \text{for all } x\in C.$$

(2) The functions are bounded.

$$\sup_{x\in C}|\,\phi_t(x)\,| \le M < \infty, \qquad \text{for all sufficiently small } t.$$

Then, let $\mathbb{D}_{\widetilde{A}}$ be the set of $f\in C_b(I)$ such that

$$\frac{P_h f(x) - f(x)}{h} \to \widetilde{A} f(x)$$

converges boundedly pointwise and $\widetilde{A} f\in C_b(I)$. The limit defines a new operator $\widetilde{A}: C_b(I)\to C_b(I)$.

This definition of the operator $(\widetilde{A},\mathbb{D}_{\widetilde{A}})$ can be used as an equivalent definition of the infinitesimal generator. It is more intuitively clear as *it shows that the infinitesimal generator defines the behavior of the diffusion in infinitesimal moments of time in future.* Recall that, the operator $(A,\mathbb{D}_A)$ is defined by Definition 5.29.

▶ **Proposition 5.33.** $A=\widetilde{A}$ *and* $\mathbb{D}_A=\mathbb{D}_{\widetilde{A}}$.

Proof. Assume that $f\in\mathbb{D}_A$. Apply Dynkin's formula with $\tau=h$, and using Fubini theorem, we have

$$\mathbb{E}_x[f(X_h)] = f(x) + \mathbb{E}_x\left[\int_0^h Af(X_t)\,dt\right] = f(x) + \int_0^h \mathbb{E}_x[Af(X_t)]\,dt.$$

Using the Dynkin's formula, we have the following arguments

$$\begin{aligned}
\widetilde{A} f &= \lim_{h\to 0}\frac{P_h f(x)-f(x)}{h} \qquad \text{(definition of } \widetilde{A}\text{)}\\
&= \lim_{h\to 0}\frac{\mathbb{E}_x[f(X_h)]-f(x)}{h} \qquad \text{(definition of } P_t\text{)}\\
&= \lim_{h\to 0}\frac{1}{h}\int_0^h \mathbb{E}_x[Af(X_t)]\,dt \qquad \text{(Dynkin's formula)}\\
&= \lim_{h\to 0}\frac{1}{h}\int_0^h P_t Af(x)\,dt\\
&= P_0 Af(x) \qquad \text{(Fundamental Theorem of Calculus and Continuity)}\\
&= Af(x) \qquad (P_0 \text{ is id}).
\end{aligned}$$

This shows that $\widetilde{A} f = Af$ for $f\in\mathbb{D}_A$ and that $\mathbb{D}_A\subset\mathbb{D}_{\widetilde{A}}$. Now let $f\in\mathbb{D}_{\widetilde{A}}$, and we want to show f

satisfies the Definition 5.29. Namely, $\widetilde{A}f = f - R_1^{-1}f$, and this equivalently as to prove $R_1(1-\widetilde{A})f = f$.

$$\begin{aligned}
R_1(1-\widetilde{A})f &= R_1 f - \lim_{h\to 0} R_1 \frac{P_h f - f}{h} \qquad \text{(DCT to pull } R_1 \text{ inside)} \\
&= R_1 f - \lim_{h\to 0} \frac{1}{h}\int_0^\infty \mathrm{e}^{-t}(P_{t+h}f - P_t f)\ \mathrm{d}t \qquad \text{(def. of } R_1) \\
&= R_1 f - \lim_{h\to 0} \frac{1}{h}\left(\int_0^\infty \mathrm{e}^{-t}P_{t+h}f\ \mathrm{d}t - \int_0^\infty \mathrm{e}^{-t}P_t f\ \mathrm{d}t\right) \\
&= R_1 f - \lim_{h\to 0} \frac{1}{h}\left(\int_h^\infty \mathrm{e}^{-s+h}P_s f\ \mathrm{d}s - \int_0^\infty \mathrm{e}^{-t}P_t f\ \mathrm{d}t\right) \\
&= R_1 f - \lim_{h\to 0} \frac{1}{h}\left(\int_h^\infty \mathrm{e}^{-t+h}P_t f\ \mathrm{d}t - \int_h^\infty \mathrm{e}^{-t}P_t f\ \mathrm{d}t - \int_0^h \mathrm{e}^{-t}P_t f\ \mathrm{d}t\right) \\
&= R_1 f - \lim_{h\to 0} \int_h^\infty \left(\frac{\mathrm{e}^{-t+h} - \mathrm{e}^{-t}}{h}\right) P_t f\ \mathrm{d}t - \lim_{h\to 0}\frac{1}{h}\int_0^h \mathrm{e}^{-t}P_t f\ \mathrm{d}t \\
&= R_1 f - \int_0^\infty \mathrm{e}^{-t}P_t f\ \mathrm{d}t + P_0 f \\
&= R_1 f - R_1 f + f = f.
\end{aligned}$$

Hence $f \in \mathbb{D}_A$ and $f - \widetilde{A}f = R_1^{-1}f$ implies that $\widetilde{A} = A$ on $\mathbb{D}_{\widetilde{A}}$ and $\mathbb{D}_{\widetilde{A}} \subset \mathbb{D}_A$. □

▶ **Theorem 5.34** (description of infinitesimal generator using speed measure)**.** *Let X be in natural scale. For a function $f \in \mathbb{D}_A$ and $x \in \operatorname{int}(I)$, we have*

$$Af(x) = \frac{\mathrm{d}}{\mathrm{d}m}\frac{\mathrm{d}}{\mathrm{d}x}f(x)$$

where m is the speed measure, in the sense that

(1) *$f'(x)$ exists except possibly on the countable set of points $\{x \in \operatorname{int}(I) \mid m(\{x\}) > 0\}$.*

(2) *If x_1 and x_2 are the points at which $f'(x)$ exists, then*

$$f'(x_2) - f'(x_1) = \int_{x_1}^{x_2} Af(y)\ m(\mathrm{d}y).$$

Proof. Let $[a,b] \subseteq I$, then $\mathbb{E}[\tau_{a,b}] < \infty$. By Dynkin's formula,

$$\mathbb{E}_x[f(X_{\tau_{a,b}})] - f(x) = \mathbb{E}_x\left[\int_0^{\tau_{a,b}} Af(X_t)\ \mathrm{d}s\right] = \int_a^b G_{a,b}(x,y)Af(y)\ m(\mathrm{d}y). \tag{5.17}$$

The final equality comes from Corollary 5.24. Since X is in natural scale,

$$\mathbb{E}_x[f(X_{\tau_{a,b}})] = f(a)\mathbb{P}_x(\tau_a < \tau_b) + f(b)\mathbb{P}_x(\tau_b < \tau_a) = f(a)\frac{b-x}{b-a} + f(b)\frac{x-a}{b-a}.$$

Hence plug this in (5.17), we have

$$f(a)\frac{b-x}{b-a} + f(b)\frac{x-a}{b-a} - f(x) = \int_a^b G_{a,b}(x,y)Af(y)\ m(\mathrm{d}y). \tag{5.18}$$

Manipulate and multiply by $(b-a)/((x-a)(b-x))$, we have

$$\frac{f(b)-f(x)}{b-x} - \frac{f(x)-f(a)}{x-a} = \int_a^b H_{a,b}(x,y)Af(y)\ m(\mathrm{d}y),$$

where $H_{a,b}(x,y)$ is defined as

$$H_{a,b} = \begin{cases} \frac{b-y}{b-x} & a \le x \le y \le b, \\ \frac{y-a}{x-a} & a \le y \le x \le b. \end{cases}$$

Let $b \downarrow x$, $a \uparrow x$, suppose left and right derivative exist. Then

$$\begin{aligned} f^+(x) - f^-(x) &= \lim_{b \downarrow x} \int_{[x,b)} \frac{b-y}{b-x} Af(y)\ m(\mathrm{d}y) + \lim_{a \uparrow x} \int_{(a,x)} \frac{y-a}{x-a} Af(y)\ m(\mathrm{d}y) \\ &= m(\{x\})Af(x) \qquad \text{(DCT)} \end{aligned}$$

This quantity is not zero only if $m(\{x\}) \neq 0$. Now, let us focused on (5.18), replace x by $x+h$ and subtract the original equation, we have

$$\begin{aligned} f(a)\left(-\frac{h}{b-a}\right) + f(b)\left(\frac{h}{b-a}\right) - (f(x+h) - f(x)) \\ = \int_a^b (G_{a,b}(x+h,y) - G_{a,b}(x,y))\, Af(y)\ m(\mathrm{d}y). \end{aligned}$$

Multiply by $(b-a)/h$, we have

$$\begin{aligned} f(b) - f(a) - \left(\frac{f(x+h)-f(x)}{h}\right)(b-a) \\ = \int_a^b \left(\frac{G_{a,b}(x+h,y) - G_{a,b}(x,y)}{h}\right)(b-a)Af(y)\ m(\mathrm{d}y). \end{aligned}$$

Taking the limit as $h \to 0$, and assume that f' exist at x, then

$$f(b) - f(a) - f'(x)(b-a) = -\int_a^x (y-a)Af(y)\ m(\mathrm{d}y) + \int_x^b (b-y)Af(y)\ m(\mathrm{d}y).$$

Choosing $x_1 < x_2$ such that f' exists at x_1 and x_2,

$$\begin{aligned} f'(x_1)(a-b) + f(b) - f(a) &= -\int_a^{x_1} (y-a)Af(y)\ m(\mathrm{d}y) + \int_{x_1}^b (b-y)Af(y)\ m(\mathrm{d}y), \\ f'(x_2)(a-b) + f(b) - f(a) &= -\int_a^{x_2} (y-a)Af(y)\ m(\mathrm{d}y) + \int_{x_2}^b (b-y)Af(y)\ m(\mathrm{d}y). \end{aligned}$$

Subtract the second equation from the first equation,

$$\begin{aligned} (f'(x_1) - f'(x_2))\,(a-b) &= \int_{x_1}^{x_2} (y-a)Af(y)\ m(\mathrm{d}y) + \int_{x_1}^{x_2} (b-y)Af(y)\ m(\mathrm{d}y) \\ &= \int_{x_1}^{x_2} (b-a)Af(y)\ m(\mathrm{d}y). \end{aligned}$$

Rearrange this,

$$f'(x_2) - f'(x_1) = \int_{x_1}^{x_2} Af(y)\ m(\mathrm{d}y).$$

□

Suppose X is not in natural scale, then let s be the scale function of X. We define the s-derivative of a function at a point x as follows

$$\frac{\mathrm{d}f}{\mathrm{d}s}(x) = \lim_{y \to x} \frac{f(y) - f(x)}{s(y) - s(x)}.$$

▶ **Theorem 5.35.** *Let X be a one-dimensional regular diffusion with scale function s. For a function $f \in \mathbb{D}_A$ and $x \in \operatorname{int}(I)$, we have*

$$Af(x) = \frac{\mathrm{d}}{\mathrm{d}m}\frac{\mathrm{d}}{\mathrm{d}s} f(x)$$

in the sense that

(1) *The s-derivative $\mathrm{d}f/\mathrm{d}s$ exists except possibly on the set $\{x \in \operatorname{int}(I) \mid m(\{x\}) > 0\}$.*

(2) *If x_1 and x_2 are the points at which the s-derivative $\mathrm{d}f/\mathrm{d}s$ exists, then*

$$\frac{\mathrm{d}f}{\mathrm{d}s}(x_2) - \frac{\mathrm{d}f}{\mathrm{d}s}(x_1) = \int_{x_1}^{x_2} Af(y)\, m(\mathrm{d}y).$$

From previous theorem, we have

$$As(x) = \frac{\mathrm{d}}{\mathrm{d}m}\frac{\mathrm{d}s(x)}{\mathrm{d}s} = \frac{\mathrm{d}}{\mathrm{d}m}1 = 0.$$

▶ **Proposition 5.36.** *If $I = [0, \infty)$, then for every $f \in \mathbb{D}_A$, we have*

$$f^+(0) = m(\{0\})A_X f(0).$$

5.6 Construction of Regular Diffusion

Previously, we have discussed that every regular one-dimensional diffusion has two important characteristics: scale function and speed measure. This section, we will prove that these two characteristics will completely defines a one-dimensional diffusion. Namely, given

(1) A state space I, which is an interval of $\mathbb{R}$.

(2) A continuous strictly increasing function s on I.

(3) A measure m on I, which is positive and locally finite on $\mathrm{int}(I)$.

One can construct a regular diffusion with the scale function s and the speed measure m. *It is sufficient to discuss construction of a regular diffusion X in natural scale with a given speed measure.* We will consider a simplified case, where m be a measure on I which is absolutely continuous with a non-negative density with respect to the Lebesgue measure, that is

$$\int_C m(\mathrm{d}x) = \int_C m'(x)\, \mathrm{d}x,$$

where $m'(x)$ is the density of the speed measure and $m'(x) \geq 0$, and C is any measurable set $C \subset I$. or equivalently,

$$m(\mathrm{d}x) = m'(x)\mathrm{d}x$$

We also assume that $m(a, b) > 0$ for any interval $(a, b) \subset I$ to ensure regularity of the diffusion.

▶ **Definition 5.37.** Let $B = (B_t)_{t\geq 0}$ be a standard Brownian motion on $I = \mathbb{R}$. For $t \geq 0$, let

$$T_t = \inf\left\{ r \geq 0 \mid \frac{1}{2}\int_0^r m'(B_s)\, \mathrm{d}s = t \right\}.$$

▶ **Theorem 5.38.** *Let $I = \mathbb{R}$ be the state space. Let $m(\mathrm{d}x)$ be absolutely continuous on I with continuous density $m'(x) > 0$. Let $B = (B_t)_{t\geq 0}$ be a standard Brownian motion and T_t be the time-changing process defined in Definition 5.37. Then $X = (X_t)_{t\geq 0}$, where $X_t = B_{T_t}$ is a regular one-dimensional diffusion in natural scale with speed measure given by*

$$m(\mathrm{d}x) = m'(x)\mathrm{d}x, \qquad x \in I.$$

The following theorem is the general result.

▶ **Theorem 5.39.** *Let $B = (B_t)_{t\geq 0}$ be a standard Brownian motion. Let m be a measure on I such that $0 < m(a, b) < \infty$ for any finite interval $(a, b) \subset \mathrm{int}(I)$. Then there exist an increasing time-changing process T_t of stopping times such that $X = (X_t)_{t\geq 0}$, where $X_t = B_{T_t}$, $t \geq 0$ is a one-dimensional regular diffusion in natural scale on I with speed measure m.*

Overview

(1) We have a scale function s, and a (speed) measure m.

(2) Using Definition 5.37, we are able to generate any one-dimensional regular diffusion X_t in natural scale as a time changing Brownian motion by Theorem 5.38.

(3) Finally, we get $Y_t = s^{-1}(X_t)$, then Y_t will have scale function s and speed measure m.

5.7 Dynkin's Diffusion

▶ **Definition 5.40** (Dynkin's condition)**.** A regular one-dimensional diffusion on I satisfies the Dynkin's conditions if there exist functions $\mu(x)$ and $\sigma^2(x)$, which are continuous on $\operatorname{int}(I) = (\ell, r)$, $\sigma^2(x) > 0$ on $\operatorname{int}(I)$ with

(1) For any $\varepsilon > 0$,
$$\lim_{h\to 0} \frac{1}{h}\mathbb{P}_x(|\, X_h - x\,| \geq \varepsilon) = 0.$$

(2) For any $\varepsilon > 0$ and $x \in \operatorname{int}(I)$,
$$\lim_{h\to 0} \frac{1}{h}\,\mathbb{E}_x[X_h - x; |\, X_h - x\,| \leq \varepsilon] = \mu(x).$$

(3) For any $\varepsilon > 0$ and $x \in \operatorname{int}(I)$,
$$\lim_{h\to 0} \frac{1}{h}\,\mathbb{E}_x[(X_h - x)^2; |\, X_h - x\,| \leq \varepsilon] = \sigma^2(x).$$

where the convergence is bounded pointwise on all finite intervals $[a, b] \subset \operatorname{int}(I)$.

Denote $C_0^2(I)$ by the set of functions with continuous second derivative on I such that
$$\{x : f(x) > 0\} \subset [a, b] \subset \operatorname{int}(I),$$
for some a, b. Denote $\operatorname{supp}(f) = \overline{\{x : f(x) > 0\}}$.

▶ **Proposition 5.41.** *Let $f \in C_0^2(I)$. Let X be a one-dimensional regular diffusion satisfying Dynkin's conditions with generator $(A, \mathbb{D}_A)$. Then $f \in \mathbb{D}_A$, and*
$$Af(x) = \mu(x)f'(x) + \frac{1}{2}\sigma^2(x)f''(x).$$

▶ **Example 5.42** (Dynkin's condition)**.** Let $X = (X_t)_{t\geq 0}$ be a Brownian motion with a drift, that is
$$X_t = X_0 + \mu t + \sigma B_t, \qquad t \geq 0,$$
where $B = (B_t)_{t\geq 0}$ is a standard Brownian motion. Then let us prove that X satisfies Dynkin's conditions. Fix $\varepsilon > 0$ and first,
$$\begin{aligned}
\frac{1}{h}\mathbb{P}_x(|\, X_h - x\,| \geq \varepsilon) &= \frac{1}{h}\mathbb{P}_x(|\, \mu h + \sigma B_h\,| \geq \varepsilon) \\
&\leq \frac{\mathbb{E}[(\sigma B_h + \mu h)^4]}{h\varepsilon^4} \qquad \text{(Markov inequality)} \\
&\leq \frac{16\,\mathbb{E}[\sigma^4 B_h^4 + (\mu h)^4]}{h\varepsilon^4} \\
&= \frac{16}{\varepsilon^4}(3\sigma^4 h + \mu^4 h^3) \to 0
\end{aligned}$$
as $h \to 0$. Hence we proved (1). Secondly,
$$\frac{1}{h}\,\mathbb{E}_x[X_h - x; |\, X_h - x\,| \leq \varepsilon] = \frac{\mathbb{E}_x[X_h - x]}{h} - \frac{\mathbb{E}_x[X_h - x; |\, X_h - x\,| > \varepsilon]}{h}.$$
The first term is
$$\frac{\mathbb{E}_x[X_h - x]}{h} = \frac{\mathbb{E}_x[\mu h + \sigma B_h]}{h} = \mu.$$
The second term can be bounded
$$\begin{aligned}
\frac{\mathbb{E}_x[X_h - x; |\, X_h - x\,| > \varepsilon]}{h} &\leq \frac{\mathbb{E}_x[(X_h - x)^4]}{\varepsilon^3 h} \\
&= \frac{\mathbb{E}_x[(\mu h + \sigma B_h)^4]}{\varepsilon^3 h} \\
&\leq \frac{16\,\mathbb{E}_x[(\mu h)^4 + \sigma^4 B_h^4]}{\varepsilon^3 h} \\
&= \frac{16}{\varepsilon^3}(3\sigma^4 h + \mu^4 h^3) \to 0
\end{aligned}$$
as $h \to 0$, and this proves (2). Condition (3) can be proved in an analogous way.

▶ **Example 5.43** (infinitesimal generator)**.**

(1) The Ornstein-Uhlenbeck has the generator

$$A = -\beta x \frac{\mathrm{d}}{\mathrm{d}x} + \frac{\sigma^2}{2} \frac{\mathrm{d}^2}{\mathrm{d}x^2}.$$

(2) Generally, if $X = (X_t)_{t \geq 0}$ solves the SDE

$$\mathrm{d}X_t = \mu(X_t)\ \mathrm{d}t + \sigma(X_t)\ \mathrm{d}B_t,$$

then under some suitable and general conditions, X *is a one-dimensional regular diffusion satisfying Dynkin's conditions with the drift coefficient* $\mu(x)$ *and diffusion coefficient* $\sigma(x)$. Then the generator of X is given by

$$A = \mu(x) \frac{\mathrm{d}}{\mathrm{d}x} + \frac{\sigma^2(x)}{2} \frac{\mathrm{d}^2}{\mathrm{d}x^2}.$$

▶ **Proposition 5.44** (construct scale function for Dynkin's diffusion)**.** *Let X be a regular one-dimensional diffusion satisfying Dynkin's conditions. Let $\mu(x)$ be the drift coefficient and $\sigma(x)$ be the diffusion coefficient. Then the scale function $s(x)$ given by (up to affine transformation) by the following expression*

$$s(x) = \int_{x_0}^{x} \exp\left(- \int_{y_0}^{y} \frac{2\mu(z)}{\sigma^2(z)}\ \mathrm{d}z \right)\ \mathrm{d}y, \qquad x \in I, \quad x_0, y_0 \in \mathrm{int}(I).$$

▶ **Example 5.45** (scale function for Brownian motion)**.** The standard Brownian motion satisfies Dynkin's conditions with $\mu(x) = 0$ and $\sigma(x) = 1$. Hence with $x_0 = y_0 = 1$, we have

$$s(x) = \int_0^x \exp\left(- \int_0^y 0\ \mathrm{d}z \right)\ \mathrm{d}y = x.$$

Having defined the scale function for Dynkin's diffusion, we then want to construct the speed measure. Let us consider the transformation $\widetilde{X} = s(X) = (s(X_t))_{t \geq 0}$ of X such that $\widetilde{X}$ is in natural scale.

▶ **Proposition 5.46** (scale function transformation of Dynkin's diffusion)**.** *Let X be a regular one-dimensional diffusion satisfying Dynkin's conditions. Assume that s is continuously differentiable. Then $\widetilde{X} = s(X)$ is a regular one-dimensional diffusion on $s(I)$ (where I is the state space of X), which satisfies Dynkin's conditions with drift $\widetilde{\mu}(y) = 0$, and diffusion coefficient $\widetilde{\sigma}(y)$ given by*

$$\widetilde{\sigma}^2(y) = \sigma^2(x) \left(s'(x)\right)^2, \qquad y = s(x) \in s(I).$$

▶ **Proposition 5.47** (arbitrary transformation of Dynkin's diffusion)**.** *Let X be a regular one-dimensional diffusion on I satisfying Dynkin's conditions with drift coefficient $\mu_X(x)$ and diffusion coefficient $\sigma_X(x)$. Let g be a function on I, which is twice continuously differentiable and $|\,g'(x)\,| \neq 0$ on $\mathrm{int}(I)$. Then $Y = g(X)$ is also a regular one-dimensional regular diffusion on $g(I)$ satisfying Dynkin's condtions with drift coefficient $\mu_Y(y)$ and diffusion coefficient $\sigma_Y^2(y)$. If $y = g(x)$, then we have*

$$\mu_Y(y) = \frac{\sigma_X^2(x) g''(x)}{2} + \mu_X(x) g'(x), \qquad \sigma_Y^2(y) = \sigma_X^2(x)(g'(x))^2.$$

▶ **Example 5.48** (application to geometric Brownian motion)**.** Let us compute the generator of the geometric Brownian motion. Let $X = (X_t)_{t \geq 0}$ be a Brownian motion with drift, that is $X_t = X_0 + \mu t + \sigma B_t$, $t \geq 0$. Then $Y = g(X_t) = \mathrm{e}^X = (\mathrm{e}^{X_t})_{t \geq 0}$ is called *geometric Brownian motion*. From Example 5.42, we have $\mu_X(x) = \mu$ and $\sigma^2(x) = \sigma^2$. Moreover, the transformation $g(x) = \mathrm{e}^x$ is infinitely many continuously differentiable, always strictly positive, and the state space of Y is $(g(-\infty), g(\infty)) = (0,\ infty)$. Let $y = g(x) = \mathrm{e}^x$, i.e. $x = \ln(y)$ for $y \in I = (0, \infty)$. By 5.47, Y is a regular one-dimensional diffusion satisfying Dynkin's conditions with

$$\mu_Y(y) = \frac{\sigma^2}{2} \mathrm{e}^x + \mu \mathrm{e}^x = \left(\mu + \frac{\sigma^2}{2} \right) y, \qquad \sigma_Y^2(y) = \sigma^2 (\mathrm{e}^x)^2 = \sigma^2 y^2.$$

Hence, the generator of the geometric Brownian motion is given by

$$A_Y = \left(\mu + \frac{\sigma^2}{2} \right) y \frac{\mathrm{d}}{\mathrm{d}y} + \frac{\sigma^2 y^2}{2} \frac{\mathrm{d}^2}{\mathrm{d}y^2}.$$

Let us now give a result on how to compute the speed measure.

▶ **Proposition 5.49** (speed measure for Dynkin's diffusion in natural scale)**.** *Let X be a regular one-dimensional diffusion in natural scale satisfying Dynkin's conditions. Let $\sigma(x)$ be the diffusion coefficient. Then the speed measure on* $\operatorname{int}(I)$ *is given by*

$$m(\mathrm{d}x) = \frac{2\ \mathrm{d}x}{\sigma^2(x)}, \qquad x \in \operatorname{int}(I).$$

▶ **Theorem 5.50.** *Let $m(\mathrm{d}x) = m'(x)\ \mathrm{d}x$ on $\operatorname{int}(I)$, where $m'(x)$ is continuous and strictly positive on $\operatorname{int}(I)$. The process in natural scale with speed measure m satisfies Dynkin's conditions with $\mu(x) = 0$ and $\sigma^2(x) = m'(x)/2$.*

5.8 Characteristic Operator and Boundary Problems

This section will link the diffusion with the differential equations. In one dimensional case, the problem will related to the ordinary differential equations (ODEs). Recall Definition 5.32 and Proposition 5.33, the definition of infinitesimal generator is

$$Af(x) = \lim_{t\to 0} \frac{\mathbb{E}_x[f(X)_t] - f(x)}{t}, \qquad f(x) \in \mathbb{D}_A.$$

Let us first discuss the notion of *characteristic operator*. For any $x \in I$, let J_n be a sequence of open neighborhoods of x converging to $J_n \downarrow x$. For example, one can take $x \in (\ell, r)$, and $J_N = (x - N^{-1}, x + N^{-1})$. Let

$$\sigma_{J_n} = \inf\{t > 0 : X_t \neq J_n\}$$

be the first time the diffusion leaves J_n.

▶ **Proposition 5.51.** *Let $f \in \mathbb{D}_A$ and x be instantaneous. Then, for any sequence J_n of bounded open neighborhoods of x converging to $J_n \downarrow x$,*

$$Af(x) = \lim_{J_n \downarrow x} \frac{\mathbb{E}_x[f(X_{\sigma_{J_n}})] - f(x)}{\mathbb{E}_x[\sigma_{J_n}]}$$

Using this proposition, we can extend the notion of infinitesimal generator.

▶ **Definition 5.52** (characteristic operator)**.** We say that f belongs to the *domain* of the *characteristic operator* and write $f \in \mathbb{D}_{\mathfrak{A}}$ if

$$\mathfrak{A}f(x) := \lim_{J_n \downarrow x} \frac{\mathbb{E}_x[f(X_{\sigma_{J_n}})] - f(x)}{\mathbb{E}_x[\sigma_{J_n}]}$$

exists for all x. We call $\mathfrak{A}$ the *characteristic operator* of the diffusion X.

▶ **Corollary 5.53.** *If $f \in \mathbb{D}_A$, then $f \in \mathbb{D}_{\mathfrak{A}}$ and $Af = \mathfrak{A}f$.*

We now connect the regular one-dimensional diffusion with boundary problems of second order differential equations. Let X be a regular one-dimensional diffusion with the infinitesimal generator $(A, \mathbb{D}_A)$. Let us define

$$u(x) = \mathbb{E}_x[F(X_{\tau_{a,b}})]$$
$$v(x) = \mathbb{E}_x\left(\int_0^{\tau_{a,b}} g(X_t)\ \mathrm{d}t\right)$$

where $F : \{a, b\} \to \mathbb{R}$ and $g : \overline{C} \to \mathbb{R}$. Then,

(1) $u(x)$ satisfies the following *Dirichlet problem*:

$$Au = 0, \qquad x \in C, \quad u(x) = F(x), \qquad x \in \{a, b\};$$

(2) $v(x)$ satisfies the following *Dirichlet-Poisson problem*:

$$Av = -g, \qquad x \in C, \quad v(x) = 0, \qquad x \in \{a, b\}.$$

▶ **Example 5.54.** Let $X = (X_t)_{t\geq 0}$ be a standard Brownian motion started at $x \in \mathbb{R}$ under $\mathbb{P}_x$. Namely, $X_t = x + B_t$, where $B = (B_t)_{t\geq 0}$ is the standard Brownian motion started at 0. Compute the function $u(x) = \mathbb{P}_x(\tau_1 < \tau_{-1})$, for $x \in [-1, 1]$.

Note that $u(x) = \mathbb{E}_x[\mathbf{1}_{\tau_1<\tau_{-1}}] = \mathbb{E}_x[F(x)]$, where $F(-1) = 0$ and $F(1) = 1$. Moreover, the infinitesimal generator of X is

$$A = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}.$$

Hence, we have $u(x)$ satisfies the *Dirichlet* problem,

$$Au(x) = 0 \implies \frac{u''(x)}{2} = 0, \qquad x \in (-1, 1)$$
$$u(-1) = 0, \quad u(1) = 1.$$

Solving this second order differential equation gives $u(x) = (x + 1)/2$ for $x \in [-1, 1]$. Notice, this is the scale function of X_t in $[-1, 1]$.

▶ **Example 5.55.** Under the same setting with the previous example, we would like to compute $v(x) = \mathbb{E}_x[\tau_{-1,1}]$. Notice that $v(x)$ can be written as

$$v(x) = \mathbb{E}_x[\tau_{-1,1}] = \mathbb{E}_x\left(\int_0^{\tau_{-1,1}} 1 \ \mathrm{d}t\right).$$

Therefore, we have the following *Dirichlet-Poisson* problem,

$$Av = -g \implies \frac{v''(x)}{2} = -1, \qquad x \in (-1, 1),$$
$$v(-1) = 0, \quad v(1) = 0.$$

The general solution gives $v(x) = 1 - x^2$.

5.9 Backward and Forward Kolmogorov Equations

▶ **Theorem 5.56** (Kolmogorov equations)**.** *Consider a regular one-dimensional diffusion X with transition operators $(P_t)_{t\geq 0}$ and a generator $(A, \mathbb{D}_A)$. Then, for all $f \in \mathbb{D}_A$, $P_t f \in \mathbb{D}_A$, the following two equation holds:*

(1) *backward Kolmogorov equation*

$$\frac{\mathrm{d}}{\mathrm{d}t}P_t f = AP_t f, \qquad t > 0, \tag{5.19}$$

(2) *forward Kolmogorov equation*

$$\frac{\mathrm{d}}{\mathrm{d}t}P_t f = P_t Af, \qquad t > 0. \tag{5.20}$$

Suppose now that X satisfies Dynkin's conditions with drift coefficient $\mu(x)$ and diffusion coefficient $\sigma(x)$. Fix $f \in C_0^2(I) \subset \mathbb{D}_A$, and denote $f(t, x) = P_t f(x)$ and recall that for Dynkin's diffusion

$$A = \mu(x)\frac{\mathrm{d}}{\mathrm{d}x} + \sigma^2(x)\frac{\mathrm{d}^2}{\mathrm{d}x^2}.$$

Then (5.19) becomes a second order differential equation

$$\frac{\partial f(t,x)}{\partial t} = AP_t f = \mu(x)\frac{\partial f(t,x)}{\partial x} + \frac{\sigma^2(x)}{2}\frac{\partial^2 f(t,x)}{\partial x^2}, \tag{5.21}$$

with the initial condition $f(0, x) = P_0 f(x) = f(x)$.

▶ **Example 5.57.** Consider the generator $A = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}$. Then, (5.21) becomes

$$\frac{\partial f(t,x)}{\partial t} = \frac{1}{2}\frac{\partial^2(t,x)}{\partial x^2}, \quad f(0,x) = f(x), \qquad x \in \mathbb{R},$$

which is the *heat equation*. The general solution for heat equation with initial condition is given by

$$f(t,x) = \frac{1}{\sqrt{2\pi t}} \int_R \exp\left(-\frac{(x-y)^2}{2t}\right) f(y)\, \mathrm{d}y$$

and

$$G(t,x,y) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(x-y)^2}{2t}\right)$$

is called *the Green function* of the heat equation. Notice that, this is a transition density of the *standard Brownian motion.*

If the transition density $p_t(x,y)$ of diffusion X exists and is sufficiently smooth then it can be found directly from the following theorem.

► **Theorem 5.58** (backward equation for densities)**.** *Let $X = (X_t)_{t\geq 0}$ be a regular one-dimensional diffusion on $\mathbb{R}$ satisfying Dynkin's conditions with bounded drift and diffusion coefficient $\mu(x)$ and $\sigma(x)$. Assume that X has transition density $p_t(x,y)$ which is continuously differentiable in $t > 0$ and twice differentiable in x, which, together with its first derivative in time and second derivative in space, is vanishing as $x \to \infty$ for all $t > 0$. Then,*

$$\frac{\partial p_t(x,y)}{\partial t} = \mu(x)\frac{\partial p_t(x,y)}{\partial x} + \frac{\sigma^2(x)}{2}\frac{\partial^2 p_t(x,y)}{\partial x^2}, \qquad t > 0, x, y \in \mathbb{R}.$$

► **Theorem 5.59** (forward equation for densities)**.** *Let $X = (X_t)_{t\geq 0}$ be a diffusion on $\mathbb{R}$ satisfying Dynkin's conditions with drift coefficient $\mu(x) \in C(\mathbb{R})$ and diffusion coefficient $\sigma^2(x) \in C^2(\mathbb{R})$. Assume that the transition density $p_t(x,y)$ of X is such that p, $\partial p/\partial t$, $\partial p/\partial y$, and $\partial^2 p/\partial^2 y \in C((0,\infty)\times\mathbb{R}\times\mathbb{R})$. Then Kolmogorov's forward equation holds,*

$$\frac{\partial p_t(x,y)}{\partial t} = -\frac{\partial\big(\mu(y)p_t(x,y)\big)}{\partial y} + \frac{1}{2}\frac{\partial^2\big(\sigma^2(y)p_t(x,y)\big)}{\partial y^2}, \qquad t > 0, x, y \in \mathbb{R}.$$