Bartels–Stewart Algorithms*

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1 Introduction to Sylvester Equation

The Sylvester equation is a linear system of the form

$$AX + XB = C, \qquad A \in \mathbb{C}^{m\times m}, \quad X, C \in \mathbb{C}^{m\times n}, \quad B \in \mathbb{C}^{n\times n}. \tag{1.1}$$

Sylvester [4] solves the homogeneous type of (1.1) in 1884. He solve the $AX = XB$ where A, X and $B \in \mathbb{R}^{n\times n}$ by considering the n^2 numbers of scalar linear equations.

The Sylvester equation also arises in eigenproblem. For instance in [1], they discover that given a block triangular matrix

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix},$$

there exists $Z = \left[\begin{smallmatrix} I & -X \\ 0 & I \end{smallmatrix}\right]$, whose inverse is simply $Z^{-1} = \left[\begin{smallmatrix} I & X \\ 0 & I \end{smallmatrix}\right]$ [1], that can block diagonalize A.

$$ZAZ^{-1} = \begin{bmatrix} I & -X \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{11} & A_{22} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} I & X \\ 0 & I \end{bmatrix} = \begin{bmatrix} A_{11} & A_{11}X - XA_{22} + A_{12} \\ 0 & A_{22} \end{bmatrix}$$

if X satisfies the Sylvester equation $A_{11}X - XA_{22} = -A_{12}$.

The Sylvester equation is solvable if and only if $\Lambda(A) \cup \Lambda(B) = \emptyset$.

2 Bartels–Stewart Algorithm

The standard algorithm for solving the Sylvester equation is using the Bartels–Stewart algorithm [2]. It computes the real Schur decompositions of A and B, $A = US_AU^*$ and $B = US_BU^*$, where U and V are unitary and S_A and S_B are quasi–upper triangular. Then the Sylvester equation (1.1) becomes

$$U^*AUU^*XV + U^*XVV^*BV = U^*CV,$$

which, by writing $Z = U^*XV$ and $D = U^*CV$, gives

$$S_AZ + ZS_B = D. \tag{2.1}$$

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Equating jth columns in (2.1), we have

$$S_A z_j + \sum_{k=1}^{j} S_{B,kj} z_k = d_j, \qquad j = 1:n,$$

where z_j and d_j are the jth column of Z and D. We can further rewrite it as

$$(S_A + S_{B,jj} I) z_j = d_j - \sum_{k=1}^{j-1} S_{B,kj} z_k, \qquad j = 1:n.$$

These n triangular systems can be solved to obtain the columns of Z can recover the solution X by $X = UZV^*$. Each triangular system is nonsingular if and only if the Sylvester equation itself is nonsingular, i.e. $\Lambda(A) \cup \Lambda(B) = \emptyset$.

We summarize this algorithm in Algorithm 1

Algorithm 1. *Bartels-Stewart algorithm for solving Sylvester equation*

Input: Coefficient matrices $A \in \mathbb{C}^{m\times m}$, $B \in \mathbb{C}^{n\times n}$ and $C \in \mathbb{C}^{m\times n}$.
Output: $X \in \mathbb{C}^{m\times n}$ that solves $AX + XB = C$.
1: Compute Schur decompositions $A = US_AU^*$, and $B = VS_BV^*$.
The algorithm can test the condition $\Lambda(A) \cup \Lambda(B) = \emptyset$ here.
2: Compute $D = U^*CV$.
3: Set $Z = [\,]$.
4: **for** $j = 1:n$ **do**
5: Solve z_j for the linear system

$$(S_A + S_{B,jj} I) z_j = d_j - \sum_{k=1}^{j-1} S_{B,kj} z_k.$$

6: Form $Z = [Z, z_j]$.
7: **end for**
8: Compute $X = UZV^*$.

Algorithm 1 costs about $25(m^3 + n^3) + 3(m^2 n + mn^2)$.

Since the algorithm is based on unitary transformations, it is numerically stable in sense that the computed $\widehat{X}$ satisfies

$$\|A\widehat{X} + \widehat{X}B - C\|_{\mathrm{F}} \le f(m,n)u(\|A\|_{\mathrm{F}} + \|B\|_{\mathrm{F}})\|\widehat{X}\|_{\mathrm{F}}, \tag{2.2}$$

where $f(m,n)$ is a modest function in m and n [3, sec. 16.1].

2.1 Instability

It is important that (2.2) does *not* means Algorithm 1 is backward stable. Since a small residual, in this case, *does not* implies a small backward error defined as

$$\eta := \min\left\{ \|[\Delta A/\|A\|_{\mathrm{F}}, \Delta B/\|B\|_{\mathrm{F}}, \Delta C/\|C\|_{\mathrm{F}}]\|_{\mathrm{F}} : (A + \Delta A)\widehat{X} + \widehat{X}(B + \Delta B) = C + \Delta C \right\}.$$

See [3, sec. 16.2]. Ideally, we should result in

$$\|A\widehat{X} + \widehat{X}B - C\|_{\mathrm{F}} \le f(m,n)\mathbf{u}\|C\|_{\mathrm{F}},$$

which indicates the numerical method is *backward stable.*

Example 2.1 ([3, p. 311])**.** The following example shows when the solution $\widehat{X}$ achieves a small residue but not a small relative error η. Let

$$A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}, \quad B = A - \alpha \begin{bmatrix} 1+\alpha & 0 \\ 0 & 1 \end{bmatrix},$$

where α is a small constant. Let $\mathrm{vec}(C)$ be the singular vector corresponding to the smallest singular value of $(I \otimes A - B^T \otimes I)$. Figure 1 reports the relative backward error and the relative residue defined as

$$\mathcal{E}_\eta := \frac{\|A\widehat{X} + \widehat{X}B - C\|_{\mathrm{F}}}{\|C\|_{\mathrm{F}}}, \quad \mathcal{E}_{\mathrm{Res}} := \frac{\|A\widehat{X} + \widehat{X}B - C\|_{\mathrm{F}}}{(\|A\|_{\mathrm{F}} + \|B\|_{\mathrm{F}})\|\widehat{X}\|_{\mathrm{F}}},$$

respectively, for different α. We observe that for different size of α, the residue error $\mathcal{E}_{\mathrm{Res}}$ remains small. However, the backward error $\mathcal{E}_\eta$ can blow up to 10^9.

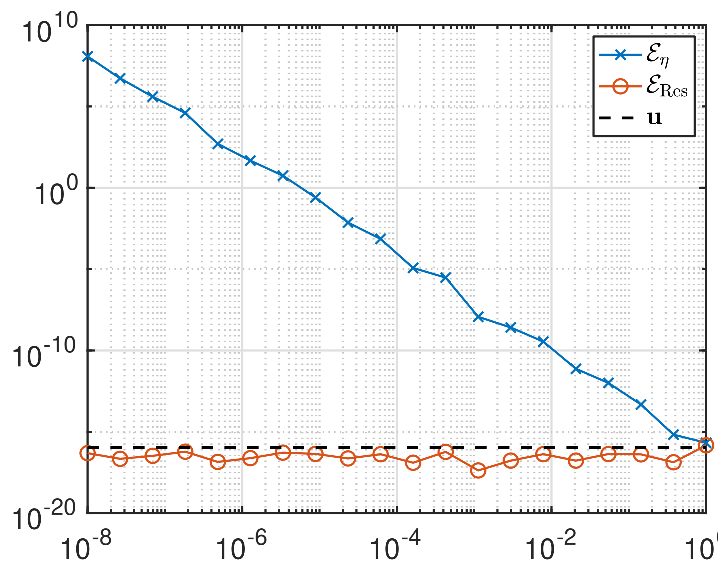

Figure 1: Behavior of $\mathcal{E}_\eta$ and $\mathcal{E}_{\mathrm{Res}}$ against different α. $\mathbf{u} = 2^{-53}$ is the unit roundoff of double precision. The working precision is double precision.

Equally effective examples are easily generated using random, ill–conditioned A and B.

Example 2.2. Changing the working precision to high precision can save some digits, but not much. If we perform Example 2.1 at quadruple precision, then $\mathcal{E}_\eta$ for $\alpha = 10^{-8}$ reduces from 10^{10} to 10^{0}, which is still large.

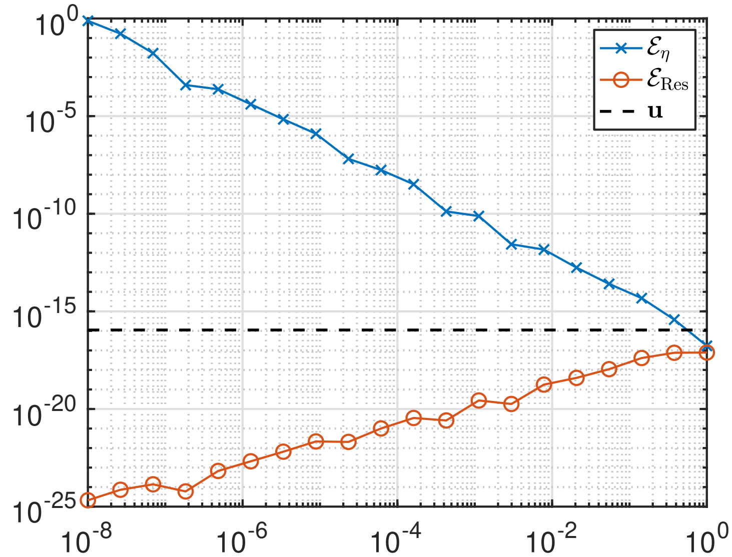

Figure 2: Behavior of $\mathcal{E}_\eta$ and $\mathcal{E}_{\text{Res}}$ against different α. $\mathbf{u} = 2^{-53}$ is the unit roundoff of double precision. The working precision is quadruple precision.

3 Implementation in MATLAB

Listing 1 shows the MATLAB implementation of the Bartels–Stewart algorithm.

```
function X = zb_sylv( A, B, C)
%ZB_SYLV - Solving the Sylvester equation

[mA,nA] = size( A );
[mB,nB] = size( B );
[mC,nC] = size( C );
if ( mA~=mC ) || ( mB~=nC )
    error("Right-hand side matrix is inconsistent" + ...
        "with coefficient matrices.");
end
if ( mA~=nA ) || ( mB~=nB )
    error("Coefficent matrices are not square.");
end
% Schur decomposition
[U,SA] = schur( A,'complex' );
[V,SB] = schur( B,'complex' );
% Compute the new right-hand side matrix
D = U'*C*V;
Z = zeros( mC,nC );
I = eye( mA,nA );
% Solve nC triangular linear system
Z( :,1 ) = ( SA + SB( 1,1 )*I )\D( :,1 );
for j = 2:nC
    sum1 = Z( :,1:j-1 ) * SB( 1:j-1,j );
    rhsj = D( :,j ) - sum1;
    Z( :,j ) = ( SA + SB( j,j )*I )\rhsj;
```

```
end
% Recover the solution
X = U*Z*V';
end
```

Listing 1: MATLAB implementation of Algorithm 1.

4 Discussion

After carefully study the mixed-precision version of Bartles-Stewart algorithm, it seems not possible to compute a backward stable solution to the Sylvester equation by using a higher precision.

Higham [3] mentioned the size of a large $\mathcal{E}_\eta$ is due to the solution $\widehat{X}$ is ill-conditioned. Then, the question remained is:

> What are the necessary and sufficient conditions for a Sylvester equation to have a backward stable solution?

References

[1] Zhaojun Bai and James Demmel. On swapping diagonal blocks in real Schur form. *Linear Algebra and its Applications*, 186:75–95, 1993. (Cited on p. 1)

[2] Richard H. Bartels and Gilbert W. Stewart. Solution of the matrix equation $AX + XB = C$. *Communications of the ACM*, 15(9):820–826, 1972. (Cited on p. 1)

[3] Nicholas J. Higham. *Accuracy and Stability of Numerical Algorithms*. Second edition, Society for Industrial and Applied Mathematics, Philadelphia, PA, USA, January 2002. xxx+680 pp. ISBN 0-89871-521-0. (Cited on pp. 2, 3, 5)

[4] James J. Sylvester. Sur l'equation en matrices $px = xq$. *Comptes Rendus de l'Académie des Science*, 115–116:67–71, 1884. (Cited on p. 1)