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Computing Accurate Eigenvalues using the Preconditioned Jacobi Algorithm

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Joint work with Nick Higham, Françoise Tisseur and Marcus Webb.

Accuracy of the Eigenvalues

$A \in \mathbb{R}^{n\times n}$ symmetric positive definite. Interested in the **relative forward error**

Tridiagonalization based methods

$$\frac{|\lambda_i(A) - \widetilde{\lambda}_i(A)|}{\lambda_i(A)} \le p(n)\, u\, \kappa(A).$$

- $p(n)$ = low deg. poly., u = working precision.
- $\kappa(A)$ = 2 norm condition number of A.
- $\lambda_i(A), \widetilde{\lambda}_i(A)$ = ith largest exact,computed eigenvalue.

"More accurate" Jacobi algorithm

$$\frac{|\lambda_i(A) - \widetilde{\lambda}_i(A)|}{\lambda_i(A)} \le p(n)\, u\, \kappa(DAD), \quad D = \mathrm{diag}(a_{ii}^{-1/2}).$$

Preconditioned Jacobi Algorithm

Drawback: $O(n^3)$ flops with a large constant.

Preconditioned Jacobi algorithm

- Orthogonal preconditioner $\widetilde{Q} \in \mathbb{R}^{n\times n}$.
- Preconditioned matrix $\widetilde{A} = \widetilde{Q}^T A \widetilde{Q}$. $\rightsquigarrow$ Computed $\widetilde{A}_{\text{comp}}$
- Apply Jacobi to $\widetilde{A}_{\text{comp}}$.
- Transform eigenvectors.

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This will provide the same accuracy, $p(n)u\kappa(DAD)$, but with significantly less rotations.

Construction of Preconditioner

Key idea: Exploiting a low precision u_ℓ.

- **Approach 1**: Orthogonalization method (**Zhou (2022), Zhang & Bai (2022)**).

$$\text{Eigenvector matrix } Q_\ell \text{ computed at } u_\ell$$
$$\downarrow$$
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- **Approach 2**: Modified tridiagonalization method (New)
  - Perform Tridiag at u_ℓ.
  - Store Householder vectors and construct transformation matrix at u. T_ℓ, Q_T
  - Apply any eigensolver to T_ℓ at u. Q_S
  - Transform Q_S at u and obtain $\widetilde{Q}$.

Reduction of Off–diagonals

Define

$$\mathrm{off}(A) = \|A - \mathrm{diag}(a_{ii})\|_F.$$

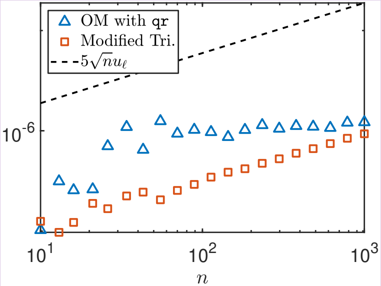
In **Zhang & Bai (2022)**:
Orthogonalization method + MGS:

$$\mathrm{off}(\widetilde{Q}^T A \widetilde{Q}) / \|A\|_F \le p(n) u_\ell.$$

We generalize this bound to cover both methods.

- $\kappa(A) = 10^8$
- Eigenvalues are geometrically distributed.
- (u_ℓ, u) = (single, double).

$\mathrm{off}(\widetilde{Q}^T A \widetilde{Q}) / \|A\|_F$



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High Precision Preconditioning

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Our Result

Given a SPD $A \in \mathbb{R}^{n \times n}$.

- Construct preconditioner $\widetilde{Q}$. (u_ℓ, u)
- Obtain preconditioned matrix $\widetilde{A} = \widetilde{Q}^T A \widetilde{Q}$. (u_h)
- Apply Jacobi to $\widetilde{A}_{\mathrm{comp}}$. (u)

$$\frac{|\widetilde{\lambda}_i(\widetilde{A}_{\mathrm{comp}}) - \lambda_i(A)|}{\lambda_i(A)} \leq p(n)u\big(\kappa(D_1 \widetilde{A} D_1) + \kappa(D_2 \widetilde{A}_{\mathrm{comp}} D_2)\big),$$

$D_1, D_2 \in \mathrm{diag}$ and $(D_1 \widetilde{A} D_1)_{ii} = (D_2 \widetilde{A}_{\mathrm{comp}} D_2)_{ii} = 1$.

High Precision Preconditioning

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Observation: For $\kappa(A) \le 1/u_\ell$, $\kappa(D_1\widetilde{A}D_1)$ and $\kappa(D_2\widetilde{A}_{\text{comp}}D_2)$ are $O(1)$.

Claim: $\kappa(D_1\widetilde{A}D_1)$ and $\kappa(D_2\widetilde{A}_{\text{comp}}D_2)$ are small compared to $\kappa(A)$ and $\kappa(DAD)$.

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Table: A spd, (u_ℓ, u, u_h) = (single, double, quadruple).

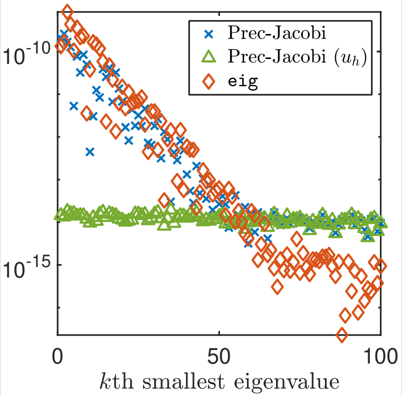
Matrix type	$\kappa(A)$	$\kappa(DAD)$	$\kappa(D_1\widetilde{A}D_1)$	$\kappa(D_2\widetilde{A}_{\text{comp}}D_2)$
`hilb(7)`	5e8	2e8	3	3
`pascal(15)`	3e15	6e12	1e4	1e4

Experiment I

Setup:

- Random matrix $A \in \mathbb{R}^{100\times100}$ SPD.
- $\kappa(A) = 10^8$.
- Geometrically distributed eigenvalues.
- (u_ℓ, u, u_h) = (single, double, quadruple).

Relative forward error

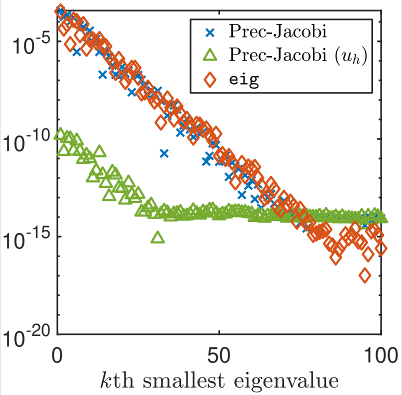


Experiment II

Setup:

- Random matrix $A \in \mathbb{R}^{100\times 100}$ SPD.
- $\kappa(A) = 10^{14}$.
- Geometrically distributed eigenvalues.
- (u_ℓ, u, u_h) = (single, double, quadruple).

Relative forward error



Summary

We proposed and analyzed:

- An alternative way to construct an preconditioner for the Jacobi algorithm.
- A modified preconditioned Jacobi algorithm with much more accurate computed eigenvalues.

$$\kappa(DAD) \to \kappa(D_1 \widetilde{A} D_1), \kappa(D_2 \widetilde{A}_{\text{comp}} D_2).$$

  The cost will be two matrix multiplications at high precision.
- Preprint in preparation.

References I

- Zhiyuan Zhang and Zheng-Jian Bai. A mixed precision Jacobi method for the symmetric eigenvalue problem. arXiv:2303.03547, November 2022.
- Zhengbo Zhou. A mixed precision eigensolver based on the Jacobi algorithm. M.Sc. thesis, The University of Manchester, Manchester, UK, September 2022.